Core Books in Advanced Mathematics

Coordinate Geometry and Complex Numbers

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Coordinate Geometry and Complex Numbers

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Preface

Advanced level mathematics syllabuses are once again undergoing changes in content and approach following the revolution in the early 1960s which led to the unfortunate dichotomy between 'modern' and 'traditional' mathematics. The current trend in syllabuses for Advanced level mathematics now being developed and published by many GCE Boards is towards an integrated approach, taking the best of the topics and approaches of modern and traditional mathematics, in an attempt to create a realistic examination target through syllabuses which are maximal for examining and minimal for teaching. In addition, resulting from a number of initiatives, core syllabuses are being developed for Advanced level mathematics consisting of techniques of pure mathematics as taught in schools and colleges at this level.

The concept of a core can be used in several ways, one of which is mentioned above, namely the idea of a core syllabus to which options such as theoretical mechanics, further pure mathematics and statistics can be added. The books in this series are core books involving a different use of the core idea. They are books on a range of topics, each of which is central to the study of Advanced level mathematics, which together cover the main areas of any single-subject mathematics syllabus at Advanced level.

Particularly at times when economic conditions make the problems of acquiring comprehensive textbooks giving complete syllabus coverage acute, schools and colleges and individual students can collect as many of the core books as they need to supplement books they already have, so that the most recent syllabuses of, for example, the London, Cambridge, AEB and JMB GCE Boards can be covered at minimum expense. Alternatively, of course, the whole set of core books gives complete syllabus coverage of single-subject Advanced level mathematics syllabuses.

The aim of each book is to develop a major topic of the single-subject syllabuses giving essential book work, worked examples and numerous exercises arising from the authors' vast experience of examining at this level. Thus, as well as using the core books in either of the above ways, they are ideal for supplementing comprehensive textbooks by providing more examples and exercises, so necessary for the preparation and revision for examinations.

In this book, we cover the requirements of the non-specialist mathematician in coordinate geometry and complex algebra in accordance with the core syllabus of pure mathematics now being included by GCE Examining Boards at Advanced level and meeting the requirements of the polytechnics and universities for entrants to degree courses in mathematics-related subjects.

In the use of coordinates, the importance of technique, that is the choice of a suitable method to tackle a problem, has been stressed. The statement and proof of standard properties of conics has been kept to a minimum, or covered by worked examples. While inevitably lacking experience, the student should try to acquire and appreciate good technique, so that more difficult problems can be tackled confidently. Only the most elementary knowledge of coordinates has been assumed, and important basic results are listed for easy reference.

In the section on complex algebra no previous knowledge is assumed; the intention is to show the usefulness of complex numbers rather than give a rigorous development of their properties from a set of axioms. On the other hand, in accordance with modern attitudes, the underlying structure of the complex field has been indicated so that the student can pursue this aspect further if desired.

Plenty of examples are provided throughout the book, both as exercises and as part of the text; the worked examples sometimes make comparisons between good and bad methods.

P. S. W. MacIlwaine
C. Plumpton

1 Basic results and techniques

1.1 Elementary results in coordinate geometry

If $P_1 \equiv (x_1, y_1)$ and $P_2 \equiv (x_2, y_2)$, we obtain the following results directly from Fig. 1.1.

Distance P_1P_2 $$\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}. \tag{1.1}$$

Gradient of P_1P_2 $$\frac{y_2 - y_1}{x_2 - x_1}. \tag{1.2}$$

Mid-point of P_1P_2 $$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right). \tag{1.3}$$

Equation of P_1P_2 $$y - y_1 = m(x - x_1), \tag{1.4}$$
$$\equiv y = mx + c,$$

where $m = (y_2 - y_1)/(x_2 - x_1)$, and $c = y_1 - mx_1$.

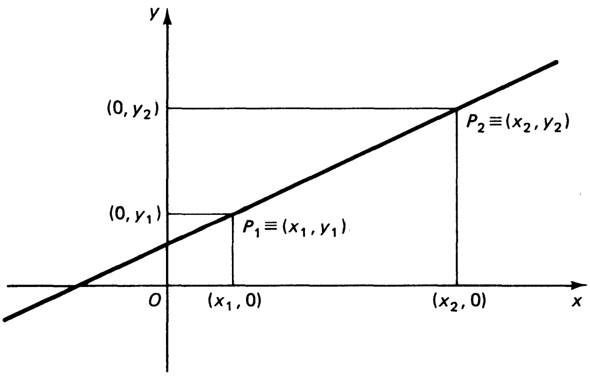

Fig. 1.1

Other important results

Straight lines with gradients m_1, m_2 are perpendicular
$$\Leftrightarrow \quad m_1 m_2 = -1. \tag{1.5}$$

Perpendicular distance of (x_1, y_1) from $ax + by + c = 0$ is
$$\frac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}}. \tag{1.6}$$

(The expression $ax_1 + by_1 + c$ is positive or negative depending on whether (x_1, y_1) is on one side of the line or the other; e.g., the points $(-2, 3)$ and $(1, 1)$ are on opposite sides of the line $5x - 2y + 1 = 0$ since $5.(-2) - 2.3 + 1 < 0$ and $5.1 - 2.1 + 1 > 0$.)

Example 1 Find an equation of that line through the meet of the lines $3x + y - 5 = 0$, $8x - 3y - 19 = 0$ which is perpendicular to the line

$$9x + 7y - 4 = 0.$$

Method (i): Solving $3x + y - 5 = 0$, $8x - 3y - 19 = 0$ we have $x = 2$, $y = -1$. Gradient of $9x + 7y - 4 = 0$ is $-9/7$; hence the gradient of a perpendicular line is $7/9$.

An equation of the required line is

$$\begin{aligned} y + 1 &= \tfrac{7}{9}(x - 2) \\ &\equiv 7x - 9y = 23. \end{aligned}$$

Method (ii): We use an important idea in coordinate geometry. The equation

$$(3x + y - 5) + \lambda(8x - 3y - 19) = 0,$$

where λ is any non-zero constant, is that of a straight line through the meet of $3x + y - 5 = 0$ and $8x - 3y - 19 = 0$, since the coordinates of this point clearly satisfy the equation for any value of λ. The gradient of this line is $(3 + 8\lambda)/(3\lambda - 1)$. Hence the line is perpendicular to $9x + 7y - 4 = 0$

$$\begin{aligned} &\Leftrightarrow \quad \frac{3 + 8\lambda}{3\lambda - 1} = \frac{7}{9} \\ &\Leftrightarrow \quad 9(3 + 8\lambda) = 7(3\lambda - 1) \\ &\Leftrightarrow \quad \lambda = -\tfrac{2}{3}. \end{aligned}$$

Hence the required line is

$$\begin{aligned} 3x + y - 5 - \tfrac{2}{3}(8x - 3y - 19) &= 0 \\ \equiv 7x - 9y &= 23. \end{aligned}$$

Exercise 1.1

1 Find an equation of the straight line which passes through the intersection of the lines

$$5x - 7y + 1 = 0, \quad 3x + 9y - 5 = 0$$

and which passes through the origin.

2 The points A and B have coordinates $(3, 0)$ and $(0, -3)$. Given that the area of the triangle ABP is $4\frac{1}{2}$ units^2, find the equations of the two lines on which P can lie.

3 Show that the equation

$$(4x + 3y - 6) + \lambda(2x - 5y - 16) = 0$$

represents a straight line for any value of λ.
Find the value of λ when the line
(a) passes through the origin,
(b) is parallel to the line $5x - 6y - 11 = 0$,
(c) is perpendicular to the line $x + 4y = 0$.

4 The points A, B, C and D have coordinates $(3, 8)$, $(2, 1)$, $(12, -4)$ and $(10, 12)$ respectively. The mid-point of CD is P.
(a) Find an equation of the line BP.
(b) The lines AC and BP meet at Q. Find the coordinates of Q.
(c) The line through B, perpendicular to AC, meets AC at R. Find the coordinates of R and the distance BR.
(d) Hence, or otherwise, calculate the area of the triangle ABQ.

5 The parallelogram $OABC$ lies in the first quadrant. The angle OAC is $90°$ and $OA = 15$. The equation of the straight line through O and A is $4y - 3x = 0$ and the equation of the straight line through O and C is $3y - 4x = 0$. Find
(a) the coordinates of A,
(b) an equation of the straight line through A and C,
(c) the coordinates of C,
(d) the coordinates of B.

6 The points $(6, 13)$, $(0, 3)$ and $(5, 0)$ are the three vertices A, B and C, respectively, of a parallelogram $ABCD$.
(a) Calculate the coordinates of D.
(b) Prove that $ABCD$ is a rectangle.
(c) Calculate the ratio of the length of AB to the length of BC.
(d) The point P lies on CD and the area of the triangle PBC is one quarter of the area of the rectangle $ABCD$. Find the coordinates of P and an equation of the line BP.

7 The points A, B, C have coordinates $(0, 2)$, $(1, 4)$ and $(3, -1)$ respectively. Given that $ABFC$ is a trapezium with CF parallel to AB and $CF = 2AB$, find
(a) the coordinates of F,
(b) an equation of the line BC,
(c) the coordinates of the point where AC cuts the x-axis,
(d) an equation of the line through B perpendicular to FC.

1.2 Loci

A locus is the path traced out by a moving point which is subject to some restriction, or, more correctly, an infinite set of points which satisfy a certain condition. For example, the locus of points in a plane equidistant from two fixed points P_1, P_2 in the plane is the perpendicular bisector (mediator) of P_1P_2. Coordinate geometry often enables us to find an equation of a locus, which we may then be able to interpret in words.

Example 2 Find an equation of the locus of points in a plane at a given distance from a fixed point in the plane (Fig. 1.2).

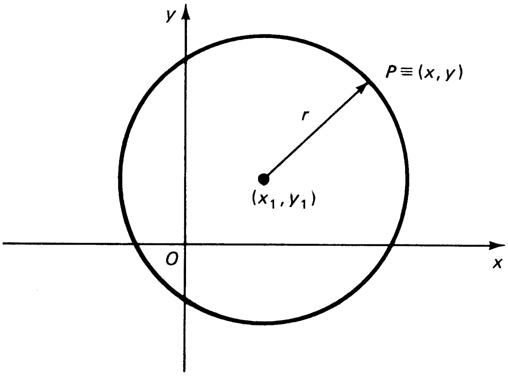

Fig. 1.2

Let (x_1, y_1) be the fixed point, r the given distance and $P \equiv (x, y)$ a point of the locus. Then, from (1.1),

$$\sqrt{[(x-x_1)^2+(y-y_1)^2]} = r$$

$$\Rightarrow \quad (x-x_1)^2+(y-y_1)^2 = r^2. \tag{1.7}$$

1.3 The equation of a circle

Defining a circle as the locus in Example 2, (1.7) is the equation of the circle with centre (x_1, y_1) and radius r. In particular, an equation of the circle whose centre is the origin and of radius a is

$$x^2+y^2 = a^2. \tag{1.8}$$

The equation

$$x^2+y^2+2gx+2fy+c = 0 \tag{1.9}$$

contains terms of the same type as (1.7), and can be written as

$$(x+g)^2+(y+f)^2 = g^2+f^2-c.$$

Hence (1.9) is an equation of the circle of centre $(-g, -f)$ and radius

$$\sqrt{(g^2+f^2-c)}$$

(provided that $g^2+f^2-c > 0$, otherwise there are no points with coordinates satisfying the equation).

It is important to notice that this result also implies its converse:

The point (x, y) lies on the circle of centre (x_1, y_1) and radius r

$$\Leftrightarrow \quad (x, y) \text{ satisfies the equation } (x-x_1)^2+(y-y_1)^2 = r^2.$$

However, the point (x, y) lies on the line $x-3y=0$

$$\Rightarrow \quad (x, y) \text{ satisfies the equation } (x-3y)(x+y) = 0,$$

but (x, y) satisfies the equation $(x - 3y)(x + y) = 0$

$$\not\Rightarrow \text{ that the point } (x, y) \text{ lies on the line } x - 3y = 0,$$

since the point may lie on the line $x + y = 0$.

Example 3 Describe the curve with equation

$$3(x^2 + y^2) - 7x + 4y - 2 = 0.$$

The equation can be rewritten

$$x^2 + y^2 - \tfrac{7}{3}x + \tfrac{4}{3}y - \tfrac{2}{3} = 0$$
$$\equiv (x - \tfrac{7}{6})^2 + (y + \tfrac{2}{3})^2 = \tfrac{2}{3} + (\tfrac{7}{6})^2 + (\tfrac{2}{3})^2 = \tfrac{89}{36}.$$

By comparison with (1.7), this equation expresses the fact that the square of the distance of (x, y) from $(7/6, -\tfrac{2}{3})$ is equal to $89/36$; hence it represents the circle of centre $(7/6, -\tfrac{2}{3})$ and radius $\tfrac{1}{6}\sqrt{89}$.

In dealing with problems involving circles, even when using coordinate methods, we should always be prepared to use the geometrical properties of a circle, which may often simplify the working. It is worth listing the more important of these properties.

(i) The perpendicular from the centre to a chord bisects the chord.

(ii) A diameter subtends a right angle at any point on the circumference.

(iii) A tangent to a circle is perpendicular to the radius at its point of contact, so that the perpendicular distance of the centre from a tangent is equal to the radius.

(iv) The point of contact of two touching circles lies on their line of centres, so that the distance between the centres is equal to either the sum (external contact) or the difference (internal contact) of the radii.

Example 4 Find an equation of the circle on the line joining $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ as diameter.

If $P(x, y)$ is a point on the circumference, the gradients of PP_1 and PP_2 are $(y - y_1)/(x - x_1)$ and $(y - y_2)/(x - x_2)$ respectively.

From property (ii) of circles $PP_1 \perp PP_2$

$$\Leftrightarrow \quad \frac{y - y_1}{x - x_1} \cdot \frac{y - y_2}{x - x_2} = -1$$

$$\Leftrightarrow \quad (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0. \tag{1.10}$$

Example 5 Find an equation of the circle through the points $(-1, 1)$ and $(3, -3)$ with its centre on the line $x - 3y + 2 = 0$.

The centre of the circle lies on the perpendicular bisector of the line joining the given points; this perpendicular is

$$y + 1 = x - 1.$$

Solving $y = x - 2$ and $x - 3y + 2 = 0$ gives the centre $(4, 2)$.
The radius equals the distance between $(4, 2)$ and $(-1, 1)$ which is

$$\sqrt{(5^2 + 1^2)} = \sqrt{26}.$$

Hence an equation of the required circle is

$$(x - 4)^2 + (y - 2)^2 = 26.$$

Example 6 Find the locus of points such that their distance from $(-3, 1)$ is twice their distance from $(0, 2)$.

If (x, y) is such a point,

$$\begin{aligned} &\sqrt{[(x+3)^2 + (y-1)^2]} = 2\sqrt{[x^2 + (y-2)^2]} \\ \Rightarrow\quad &(x+3)^2 + (y-1)^2 = 4[x^2 + (y-2)^2] \\ \Leftrightarrow\quad &3(x^2 + y^2) - 6x - 14y + 6 = 0 \\ \Leftrightarrow\quad &(x-1)^2 + (y - \tfrac{7}{3})^2 = \tfrac{40}{9}. \end{aligned}$$

This is an equation of the circle of centre $(1, 7/3)$ and radius $\frac{2}{3}\sqrt{10}$. We can clearly *generalise* this result by saying that if A and B are fixed points, the locus of a point B such that $PA = k\,.\,PB$, where k is a positive constant other than 1, is a circle.

Example 7 Find the equations of the tangents to the circle

$$x^2 + y^2 + 4x - 2y - 47 = 0$$

which are parallel to $2x - 3y + 7 = 0$.

Method (i): Any line parallel to $2x - 3y + 7 = 0$ has an equation of the form $2x - 3y + c = 0$.
Solving with $x^2 + y^2 + 4x - 2y - 47 = 0$,

$$\begin{aligned} &x^2 + 4x + \left(\frac{2x + c}{3}\right)^2 - 2\left(\frac{2x + c}{3}\right) - 47 = 0 \\ \Leftrightarrow\quad &13x^2 + 2x(2c + 12) + c^2 - 6c - 423 = 0. \end{aligned}$$

If the line is to touch the circle, this quadratic in x must have equal roots

$$\begin{aligned} \Leftrightarrow\quad &4(c + 6)^2 = 13(c^2 - 6c - 423) \\ \Leftrightarrow\quad &c^2 - 14c - 627 = 0 \quad\Leftrightarrow\quad (c + 19)(c - 33) = 0 \\ \Leftrightarrow\quad &c = -19 \text{ or } 33. \end{aligned}$$

Hence the equations of the tangents are $2x - 3y - 19 = 0$, $2x - 3y + 33 = 0$. This purely algebraic approach is lengthy and cumbersome. Use of geometry gives a shorter and more elegant solution.

Method (ii): If a line touches a circle, the perpendicular distance of the centre from the line must equal the radius.

$$x^2 + y^2 + 4x - 2y - 47 = 0 \equiv (x+2)^2 + (y-1)^2 = 52,$$

so that the circle has centre $(-2, 1)$ and radius $2\sqrt{13}$. Hence by (1.6)

$$\frac{2.(-2) - 3.1 + c}{\sqrt{(2^2 + 3^2)}} = \pm 2\sqrt{13}$$

$$\Leftrightarrow \quad c - 7 = \pm 26 \quad \Leftrightarrow \quad c = -19 \text{ or } 33 \text{ as before.}$$

Exercise 1.3

1 Show that the point $(2 + 2\cos\theta, 2\sin\theta)$ lies on the circle $x^2 + y^2 = 4x$. Find the equation of the tangent to the circle at this point.

The tangents at the points P and Q on this circle touch the circle $x^2 + y^2 = 1$ at the points R and S. Obtain the coordinates of the point of intersection of these tangents. Obtain also an equation of the circle through the points P, Q, R and S.

2 A square is inscribed in the circle $x^2 + y^2 + 4x - 2y + 1 = 0$. Find the area of the square.

Find also equations of the tangents to the circle from the origin.

3 The points A, B and C have coordinates $(4, 4)$, $(-4, 0)$ and $(6, 0)$ respectively.

(a) Find an equation of the circle through the points A, B, C.

(b) Find the coordinates of the point where the internal bisector of the angle BAC meets the x-axis.

(c) Find an equation of the circle which passes through B and touches AC at C.

4 Prove that the point $B\,(1, 0)$ is the mirror-image in the line $2x + 3y = 15$ of the point $A\,(5, 6)$.

Find an equation of

(a) the circle on AB as diameter,

(b) the circle which passes through A and B and touches the x-axis.

5 Find an equation of the circle which passes through the points $(1, 1)$, $(1, 7)$ and $(8, 8)$.

Find also equations of the tangents from the origin to this circle.

6 Two circles, which both touch both the x-axis and the line $3x - 4y + 3 = 0$, have their centres on the line $x + y = 3$. Show that an equation of one of these circles is

$$x^2 + y^2 - 4x - 2y + 4 = 0$$

and find an equation of the other.

Find also an equation of the second tangent from the origin to the given circle.

7 Given that the tangents from O touch the circle $x^2 + y^2 - 8x - 4y + 10 = 0$ at A and B, find an equation of the circle OAB and an equation of the line AB.

8 Find an equation of the perpendicular bisector of the line segment joining the points $Q(7, 6)$ and $R(-5, 0)$. This bisector meets the y-axis at S. Find the coordinates of the point P in which this bisector meets the line through the point R perpendicular to RS.

Show that the circle through Q, R and S passes through P, and find the coordinates of its centre.

9 Find an equation of the tangent to the circle $x^2 + y^2 = a^2$ at the point $T(a\cos\theta, a\sin\theta)$. This tangent meets the line $x = -a$ at R, and RT is produced to P so that $RT = TP$. Find the coordinates of P in terms of θ. Find also the coordinates of the points in which the locus of P meets the y-axis.

10 The points A and B have coordinates $(4, 3)$ and $(6/5, 17/5)$ respectively. The line through O perpendicular to AB and the line through A perpendicular to OB meet at

H. Show that the coordinates of H are $(3/5, 21/5)$. Given that BH meets OA at D, find the coordinates of D.

Write down an equation of the circle on OB as diameter and verify that this circle passes through D.

11 Prove that the point $P(5, 6)$ is the mirror-image of the point $Q(1, 0)$ in the line $2x + 3y = 15$.

Find an equation of

(a) the circle on PQ as diameter

(b) the circle which passes through P and Q and touches the x-axis.

12 Any straight line is drawn through the point $(3, -2)$ and K is the foot of the perpendicular to it from the point $(-2, 1)$. Find an equation of the locus of K and state what curve it represents.

1.4 Parametric equations

A *parameter* is a variable, denoted by t or any suitable letter, such that the coordinates (x, y) of a point on a curve can be expressed in the form $x = f(t)$, $y = g(t)$, giving *parametric equations* or *parametric coordinates* for the curve; for example, parametric equations for the circle (1.8) are $x = a\cos\theta$, $y = a\sin\theta$, since $a^2(\cos^2\theta + \sin^2\theta) = a^2$ for all θ.

Parametric equations are not usually unique; there is no general method of finding them for a particular curve. Some curves have very simple parametric equations, but eliminating the parameter may give a very complicated x, y equation; for example, the cycloid has parametric equations $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$. The advantage of parametric equations is that we use a single variable to represent a point rather than two variables related by the cartesian equation of the curve, which frequently makes for easier reference and simpler working.

Example 8 Find the gradient of the circle $x^2 + y^2 = a^2$ at a general point.

$$x^2 + y^2 = a^2 \quad \Rightarrow \quad 2x + 2y\frac{dy}{dx} = 0 \quad \Leftrightarrow \quad \frac{dy}{dx} = -\frac{x}{y}.$$

Hence the gradient at $(x_1, y_1) = -\dfrac{x_1}{y_1}$, where $x_1{}^2 + y_1{}^2 = a^2$. But using parametric equations $x = a\cos\theta$, $y = a\sin\theta$,

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = a\cos\theta \div (-a\sin\theta) = -\cot\theta.$$

We say that the gradient at the point 'θ' is $-\cot\theta$.

Example 9 Find an equation of the tangent to the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ at the point $\theta = \frac{1}{2}\pi$.

$$\begin{aligned}\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{a\sin\theta}{a(1 - \cos\theta)} &= \frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2} \\ &= \cot\theta/2 \\ &= 1 \text{ when } \theta = \tfrac{1}{2}\pi.\end{aligned}$$

$$\text{Equation of tangent} \equiv y - a(1-0) = x - a(\tfrac{1}{2}\pi - 1)$$
$$\equiv y = x + a(2 - \tfrac{1}{2}\pi).$$

Example 10 (This example can be deferred until after reading section 2.1. A curve is given parametically by the equations

$$x = a(2+t^2), \quad y = 2at, \quad a > 0.$$

Find the values of the parameter t at the points P and Q where this curve is cut by the circle with centre $(3a, 0)$ and radius $5a$.

Show that the tangents to the curve at P and Q meet on the circle, and that the normals to the curve at P and Q also meet on the circle.

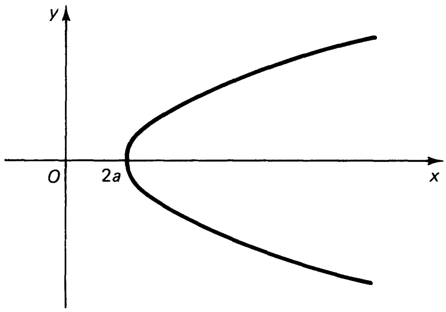

Fig. 1.3 The parabola $x = a(2+t^2)$, $y = 2at$, $a > 0$

By comparison with $x = at^2$, $y = 2at$, we see that the given curve is a parabola with vertex $(2a, 0)$, Fig. 1.3 (see Chapter 2). The circle and the parabola are both symmetrical about the x-axis; hence the tangents at P and Q and also the normals at P and Q meet on the x-axis.

An equation of the circle is $(x-3a)^2 + y^2 = 25a^2$. If P (or Q) has parameter t, then

$$(2a + at^2 - 3a)^2 + (2at)^2 = 25a^2$$
$$\Leftrightarrow \quad a^2(t^2-1)^2 + 4a^2t^2 = 25a^2$$
$$\Leftrightarrow \quad t^2 + 1 = \pm 5 \quad \Rightarrow \quad t^2 = 4 \quad (t^2+1>0)$$
$$\Leftrightarrow \quad t = \pm 2.$$

The gradient of the tangent at 't' is $1/t$. Hence an equation of the tangent at P (or Q), where $t = 2$,

$$\equiv y - 4a = \tfrac{1}{2}(x - 2a - 4a)$$
$$\equiv 2y = x + 2a.$$

This meets Ox at $(-2a, 0)$, which lies on the circle. Hence the tangents at P and Q meet on the circle at A.

If the normal at P meets the circle at B, $A\hat{P}B = 90°$. Hence AB is a diameter of the circle; thus the normals at P and Q also meet on the circle.

Miscellaneous exercise 1

1 Find an equation of the line

(a) joining the point $(2, -1)$ to the meet of the lines

$$3x - y + 7 = 0 \quad \text{and} \quad 10x - 7y - 13 = 0;$$

(b) with gradient 2 and passing through the meet of

$$x - 2y + 2 = 0 \quad \text{and} \quad 3x + y - 1 = 0;$$

(c) through the point $(3, -4)$ and the foot of the perpendicular from $(0, 2)$ to $5x + 2y - 1 = 0$.

2 Show that, for all values of k, the equation

$$(2x - y - 3) + k(x - y - 1) = 0$$

represents a line passing through the point P of intersection of the lines

$$2x - y - 3 = 0, \quad x - y - 1 = 0.$$

Show also that P lies outside the circle

$$x^2 + y^2 + 4x - 6y + 11 = 0$$

and find the values of k for which the line is a tangent to the circle. Obtain equations of the two tangents from P to the circle.

3 Find in each case the locus of the point P under the following conditions:

(a) equidistant from the points $(0, 5)$ and $(-3, 0)$;

(b) distance from the point $(-4, 2)$ is three times its distance from the point $(-1, 6)$;

(c) distance from the point $(4, 0)$ equals its distance from the line $x + 2 = 0$.

4 Find (a) an equation of the tangent at the point 't', (b) a cartesian equation of the curves with parametric equations

(i) $x = t^2 - 1$, $y = 3t$; (ii) $x = 2t - 3$, $y = 4 + t$;

(iii) $x = 2t^2$, $y = 4t^3$; (iv) $x = (1 + t)/(1 - t)$, $y = t/(1 - t)$.

5 The circle S_1 with centre C_1 (a_1, b_1) and radius r_1 touches externally, at the point P, the circle S_2 with centre C_2 (a_2, b_2) and radius r_2. The tangent at P passes through the origin. Show that

$$({a_1}^2 - {a_2}^2) + ({b_1}^2 - {b_2}^2) = ({r_1}^2 - {r_2}^2).$$

If, also, the other two tangents from the origin to S_1 and S_2 are perpendicular, prove that

$$|a_2 b_1 - a_1 b_2| = |a_1 a_2 + b_1 b_2|.$$

Deduce that, if C_1 remains fixed but S_1 and S_2 vary, C_2 lies on the curve

$$({a_1}^2 - {b_1}^2)(x^2 - y^2) + 4a_1 b_1 xy = 0.$$

6 Find an equation of the circle S which passes through the points $A(0, 4)$ and $B(8, 0)$ and has its centre on the x-axis. Given that the point C lies on the circumference of S, find the greatest possible area of the triangle ABC.

7 A triangle ABC lies wholly within the first quadrant and has an area $4\frac{1}{2}$ units^2. An equation of one side is $2x - 5y + 23 = 0$ and the vertices A and B have coordinates

(1, 5) and (3, 4) respectively. Find equations of the other two sides, the angle BAC, and the coordinates of the orthocentre of the triangle.

8 Prove that the points $A(-2, -1)$, $B(4, 3)$, $C(6, 0)$ and $D(0, -4)$ are the vertices of a rectangle.

The line $x = 3$ meets the sides AB, DC in P, Q respectively. Find the area of the trapezium $PBCQ$.

9 Prove that an equation of the straight line AB which makes intercepts a and b on the x- and y-axes respectively is

$$x/a + y/b = 1.$$

Write down an equation of the line CD whose intercepts on Ox and Oy are a^2/h and b^2/k respectively. Find the coordinates of Q, the point of intersection of AB and CD.

Prove that, if the point (h, k) lies on AB, an equation of the line joining Q to the origin is

$$kx + hy = 0.$$

10 The points $P(4, 1)$ and $Q(2, -3)$ are two vertices of an equilateral triangle PQR. Find the coordinates of the two possible positions of R.

11 Prove that the lines $x - 2y - 1 = 0$, $2x + y + 3 = 0$ are perpendicular.

These two lines are sides of a rectangle whose other sides intersect at the point (3, 4). Find equations of these other sides and the area of the rectangle.

12 Equations of two sides of a parallelogram, one of whose vertices is at the origin, are

$$5x - 12y + 66 = 0, \quad 4x - 3y - 33 = 0.$$

Find the lengths of the sides of the parallelogram.

13 A line cuts the x- and y-axes at A and B. The coordinates of the mid-point of AB are $(3, -2)$. Find the coordinates of the point of intersection of this line with the line $2x + 3y + 3 = 0$.

A variable line passes through the fixed point (h, k) and meets Ox, Oy at P and Q respectively. Prove that the locus of the mid-point of PQ is

$$\frac{h}{x} + \frac{k}{y} = 2.$$

14 Equations of the sides of a triangle ABC are

$$x + y - 4 = 0, \quad x - y - 4 = 0, \quad 2x + y - 5 = 0.$$

Prove that, for all numerical values of p and q, the equation

$$p(x + y - 4)(2x + y - 5) + q(x - y - 4)(2x + y - 5) = (x - y - 4)(x + y - 4)$$

represents a curve passing through A, B and C.

Find the values of p and q for which this curve is a circle, and hence find the centre and radius of the circumcircle of the triangle.

15 Write down the equation of the line CD whose intercepts on Ox and Oy are ma and nb respectively. Find the coordinates of P, the point of intersection of AB and CD.

Prove that, if a and b are fixed, but m and n vary so that

$$m + n = 2mn,$$

then P is a fixed point.

16 Two of the points of intersection of the curves given parametrically by

$$x = 2k\cos t, \quad y = k\sin t,$$

$$x = cs, \quad y = c/s,$$

where k and c are both positive, lie on the straight line $x = y$. Express c in terms of k.

Prove that the four points of intersection of the two curves lie at the vertices of a parallelogram whose sides have lengths k and $3k$.

2 Conics

2.1 Conics

Conic sections, or conics, are those curves that can be obtained as sections of a double right-circular cone and a plane (see Fig. 2.1). Such a section is either (i) a circle, (ii) an ellipse, (iii) a parabola, (iv) a hyperbola, or (v) a line-pair; but apart from explaining the name 'conic' we shall not use this definition.

The advantages of coordinate geometry are well illustrated by its application to a study of conics, which has great importance in science and engineering. The equation of a conic is easily recognised, being of the second degree in x and y. The most general equation of a conic is of the form

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0,$$

where a, b, c, f, g, h are constants. However, by a suitable choice of axes, the equation of any particular conic can be greatly simplified.

An important property of the ellipse, parabola and hyperbola provides a definition of these three types of conic in terms of a fixed point (*focus*) and a fixed straight line (*directrix*) as the locus of points such that the ratio of their distances from a focus and from a directrix is constant. This constant, the *eccentricity* (e), is less than, equal to, or greater than one for an ellipse, a parabola or a hyperbola, respectively. In Fig. 2.2, $SP = ePM$ with (i) $e < 1$, (ii) $e = 1$, (iii) $e > 1$.

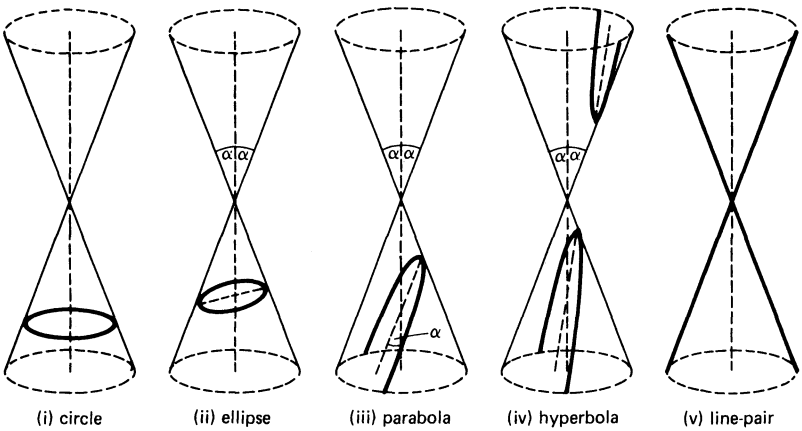

Fig. 2.1 Sections of a right-circular cone

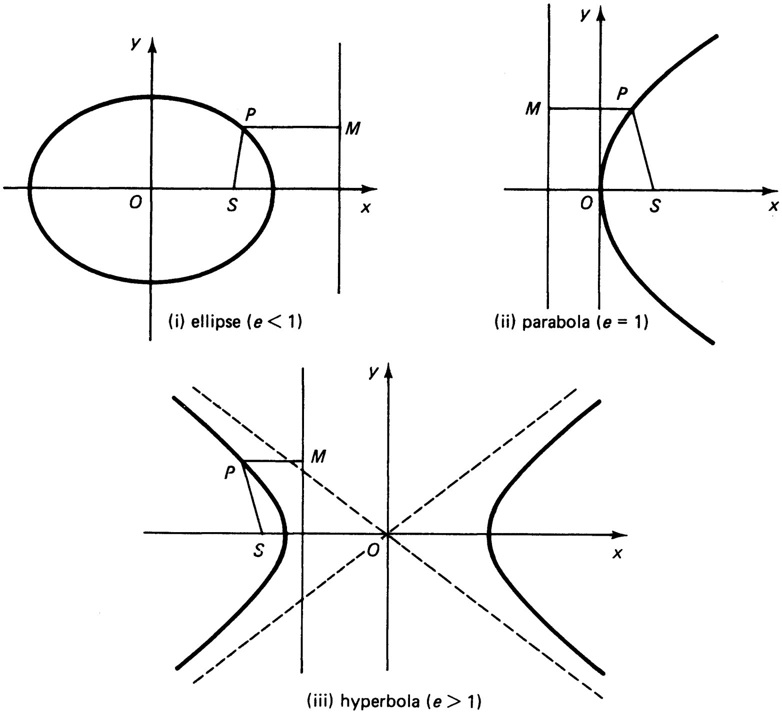

Fig. 2.2

2.2 The parabola

The equation of a parabola is usually taken as $y^2 = 4ax$ ($a > 0$), so that the curve is symmetrical about the x-axis and meets its axis of symmetry at the vertex O. Parametric equations are $x = at^2$, $y = 2at$. The focus is $(a, 0)$ and the directrix is $x = -a$.

The chord joining 't_1' and 't_2', the points P_1 and P_2, has gradient

$$\frac{2a(t_1 - t_2)}{a(t_1{}^2 - t_2{}^2)} = \frac{2}{t_1 + t_2}$$

$$\Rightarrow \quad P_1P_2 \equiv y - 2at_1 = \frac{2}{t_1 + t_2}(x - at_1{}^2)$$

$$\equiv 2x - y(t_1 + t_2) + 2at_1t_2 = 0. \tag{2.1}$$

[Note the symmetry in t_1 and t_2 of this equation. This is to be expected since the choice of t_1 or t_2 for P_1 and P_2 is of course arbitrary.]

When $t_1 = t_2 = t$, we obtain the tangent at 't'

$$\equiv y = \frac{x}{t} + at. \tag{2.2}$$

This can be obtained directly by using $\dfrac{dy}{dx} = \dfrac{2a}{2at} = \dfrac{1}{t}$, and so on.

Similarly, the normal at 't'

$$\equiv (y - 2at) = -t(x - at^2)$$

$$\equiv y + tx = 2at + at^3. \tag{2.2a}$$

Example 1 Prove that the tangents at the ends of a focal chord PQ meet at right angles on the directrix of the parabola. [A focal chord is a chord through the focus.]

Method (i): If $P(x_1, y_1)$ is a point on $y^2 = 4ax$ and S is the focus,

$$PS \equiv y = \frac{y_1}{x_1 - a}(x - a).$$

PS meets $y^2 = 4ax$ where

$$y(x_1 - a) + ay_1 = y_1 \cdot \frac{y^2}{4a}.$$

Using $x_1 = {y_1}^2/4a$

$$\Rightarrow \quad (y - y_1)(y_1 y + 4a^2) = 0$$

$$\Rightarrow \quad Q \equiv \left(\frac{a^2}{x_1}, -\frac{4a^2}{y_1}\right).$$

Tangent at $P \equiv y - y_1 = \dfrac{2a}{y_1}(x - x_1)$; $\quad \left(\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2a}{y}\right)$

tangent at $Q \equiv y + \dfrac{4a^2}{y_1} = -\dfrac{y_1}{2a}\left(x - \dfrac{a^2}{x_1}\right)$.

Comparing gradients shows that the tangents are perpendicular. Solving the equations for x, we eventually (using ${y_1}^2 = 4ax_1$) obtain $x = -a$, showing that the tangents meet on the directrix.

This method is bad because (a) we have not used parametric equations, (b) we have treated P and Q asymmetrically.

Method (ii): Obtain or write down (2.1) for PQ.

$$2x - y(t_1 + t_2) + 2at_1 t_2 = 0$$

goes through $(a, 0)$ $\Leftrightarrow$ $t_1 t_2 = -1$. Hence the tangents at P and Q are perpendicular.

The tangents $y = x/t_1 + at_1$, $y = x/t_2 + at_2$ meet where

$$x\left(\frac{1}{t_1} - \frac{1}{t_2}\right) + a(t_1 - t_2) = 0$$

$$\Rightarrow \quad x = at_1 t_2 = -a.$$

Hence the tangents meet on the directrix.

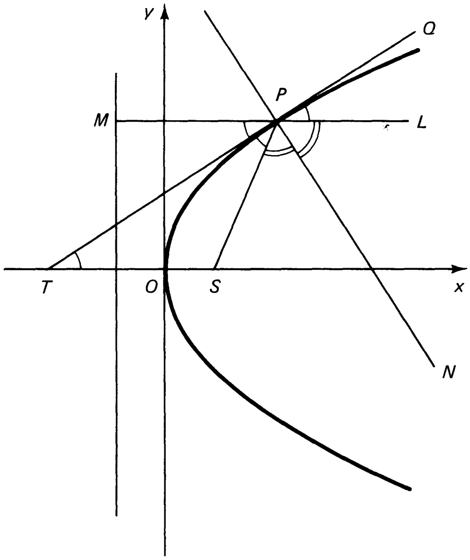

Fig. 2.3

Example 2 With reference to Fig. 2.3, prove that, for any point P on the parabola, SP and PM make equal angles with the normal PN at P.

$$PT \equiv y = \frac{x}{t} + at \tag{2.2}$$

$$\Rightarrow \quad T \equiv (-at^2, 0) \quad \Rightarrow \quad ST = at^2 + a = PM = SP$$

$$\Rightarrow \quad S\hat{P}T = S\hat{T}P = T\hat{P}M = Q\hat{P}L$$

$$\Rightarrow \quad S\hat{P}N = L\hat{P}N.$$

This is the 'reflection property' of a parabola which makes the curve so important in many practical applications. If a source of radiation, such as light, heat, or any form of electromagnetic energy, is placed at the focus of a parabolic reflector, the radiation in the plane of the parabola will be reflected parallel to the axis giving a directed beam. In practice a paraboloid, such as the reflector of a car headlamp, has a parabolic cross-section parallel to its axis and a circular section perpendicular to the axis.

Example 3 Prove that the circle on a focal chord of a parabola as diameter touches the directrix.

Plan of solution: We know from Example 1 that the tangents at the ends of a focal chord are perpendicular and meet on the directrix, and hence the point T where they meet is a point where the circle on PQ as diameter *meets* the directrix (see §1.3(ii), p. 5); but we still have to prove that the circle *touches* the directrix at T.

Since a tangent to a circle is perpendicular to its radius, the circle touches the directrix

$\Leftrightarrow$ TM is parallel to Ox, where M is the mid-point of PQ

$\Leftrightarrow$ T and M have the same y-coordinate.

Hence we can say:

solving $\quad y = \dfrac{x}{t_1} + at_1$ and $y = \dfrac{x}{t_2} + at_2, \quad y = a(t_1 + t_2)$.

But for the mid-point of PQ,

$$y = a(t_1 + t_2). \tag{1.3}$$

Therefore the circle on PQ as diameter touches the directrix.

Example 4 Find the coordinates of the point t' at which the normal to the parabola $y^2 = 4ax$ meets the curve again. Hence find the locus of mid-points of normal chords of the parabola.

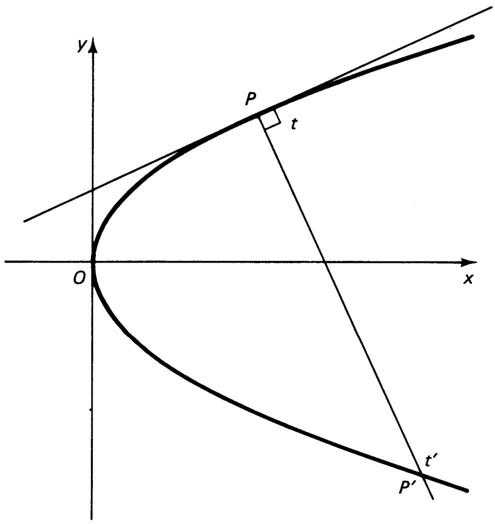

Fig. 2.4

The gradient of the normal at t is $-t$, and the gradient of the chord PP' is $2/(t + t')$ (see Fig. 2.4). Hence, for a normal chord,

$$-t = \frac{2}{t + t'} \quad \Leftrightarrow \quad t' = -\frac{t^2 + 2}{t}$$

$$\Rightarrow \quad P' \equiv \left[\frac{a(t^2 + 2)^2}{t^2}, -\frac{2a(t^2 + 2)}{t}\right].$$

For the mid-point of PP',

$$x = \tfrac{1}{2}a\left[t^2 + \frac{(t^2+2)^2}{t^2}\right] = a\left(t^2 + 2 + \frac{2}{t^2}\right),$$

$$y = a\left[t - \frac{t^2+2}{t}\right] = -\frac{2a}{t}.$$

To obtain the locus, substitute $t = -2a/y$ into the expression for x.

$$\text{Locus of mid-points} \equiv x = a\left(\frac{4a^2}{y^2} + 2 + \frac{2y^2}{4a^2}\right)$$

$$\equiv 2axy^2 = y^4 + 4a^2y^2 + 8a^4.$$

Example 5 Prove that if the normals at two variable points of a parabola meet on the curve, then the line joining the two points passes through a fixed point.

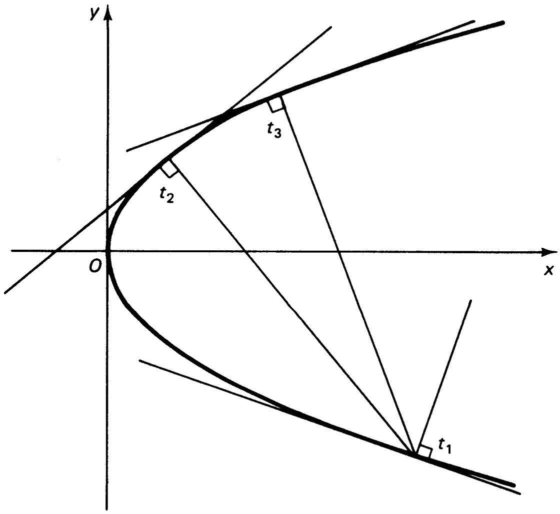

Fig. 2.5

The normal at t on the parabola is, from (2.2a),

$$y + tx = 2at + at^3.$$

This line goes through (h, k)

$$\Leftrightarrow \quad at^3 + t(2a - h) - k = 0.$$

The roots of this cubic in t, supposed all real, are the parameters t_1, t_2 and t_3 of the feet of normals through (h, k). Since two of the normals, say at t_2 and t_3, meet on the parabola, the point (h, k) must coincide with the point t_1 (see Fig. 2.5). Hence $k = 2at_1$

$$\Leftrightarrow \quad t_1 = k/2a.$$

The product of the roots of the cubic equation $px^3 + qx^2 + rx + s = 0$ is $-s/p$; hence

$$t_1 t_2 t_3 = k/a = 2t_1 \quad \Leftrightarrow \quad t_2 t_3 = 2.$$

$$\begin{aligned} \text{Chord } P_2P_3 &\equiv 2x - y(t_2 + t_3) + 2at_2t_3 = 0 \\ &\equiv 2x - y(t_2 + t_3) + 4a = 0. \end{aligned}$$

Hence the chord P_2P_3, as t_2 and t_3 vary, passes through the fixed point $(-2a, 0)$.

Exercise 2.2

In this exercise, unless otherwise stated, references are to the parabola $y^2 = 4ax$, $a > 0$, and the equations of the tangent and normal ((2.2) and (2.2a)) at the point $(at^2, 2at)$ should be used where required.

1 Prove that the tangents to the parabola at the points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ meet at the point T which has coordinates $(apq, a(p + q))$.

Note: In Questions **2–5** the notation and result of Question **1** are to be used.

2 The tangents at the ends of a variable chord PQ of the parabola meet in T.
(a) Given that the direction of PQ is fixed, prove that the point T lies on a fixed straight line.
(b) Given that the mid-point of PQ lies on a fixed straight line perpendicular to the axis of the parabola, prove that the point T lies on another fixed parabola.

3 Find the locus of T
(a) given that the chord PQ always passes through the point (a, a),
(b) given that the chord PQ always touches the parabola $y^2 = 2ax$.

4 The point N is the intersection of the normals at P and Q. Given that T lies on the line $x + 2a = 0$, show that N lies on the parabola with equation $y^2 = 4a(x - 4a)$.

5 The line TM is parallel to the axis of the parabola and meets the line PQ at M. Prove that M is the mid-point of PQ.
Given that the parameters of the points P and Q are t and $2t$ respectively, show that T always lies on the parabola $y^2 = 9ax$.

6 The tangents to the parabola from a point A on the line $x + 4a = 0$ touch the curve at P and Q. Prove that $P\hat{O}Q = 90°$.
Given that A varies on the line $x + 4a = 0$, prove that the least area of the triangle OPQ is $16a^2$.

7 The points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ are such that the tangent to the parabola at P is parallel to the chord OQ, where O is the origin. Show that $q = 2p$.
The tangents to the parabola at P and Q meet at the point T. Find the coordinates of T in terms of a and p. Find an equation of the perpendicular bisector of the line PQ, and obtain an expression, in terms of a and p, for the perpendicular distance of T from this bisector.

8 Show that the locus of points of intersection of perpendicular normals to the parabola is $y^2 = a(x - 3a)$.

9 If the normal at $(at^2, 2at)$ meets the parabola again at the point $(aT^2, 2aT)$ show that

$$t^2 + tT + 2 = 0.$$

Deduce that T^2 cannot be less than 8.
The line $3y = 2x + 4a$ meets the parabola at the points A and B. Show that the normals at A and B meet on the parabola.

10 P is the point $(ap^2, 2ap)$. The tangent to the parabola at P meets Ox at T, and M is the mid-point of OP. The line TM meets Oy at K.
(a) Prove that $TK : KM = 2 : 1$.
(b) Find the coordinates of the mid-point L of the line TM.
(c) Prove that $K\hat{O}L = K\hat{O}M$.
(d) If p varies, find an equation of the locus of L.

11 The points P $(ap^2, 2ap)$ and Q $(aq^2, 2aq)$ on the parabola are such that PO and PQ are equally inclined to Ox. Prove that $q = -2p$. Hence show that the circumcentre of the triangle OPQ lies on the normal to the parabola at P.

12 P is the point $(at^2, 2at)$ on the parabola and OQ is the chord parallel to the tangent at P. Find the coordinates of the point of intersection R of the tangents at P and Q. Explain why the locus of R is another parabola.

Given that S is the mid-point of OQ, prove that PSR is a right-angled triangle. Prove also that the area of triangle PQR is $\frac{1}{2}a^2t^3$.

13 The tangent and normal to the parabola at the point P meet Ox at T and G respectively. Find a cartesian equation of the locus of the centroid of the triangle PGT as P varies.

14 The tangent to the parabola S_1 with equation $y^2 = 4ax$ cuts Ox, Oy at P and Q respectively, and M is the mid-point of PQ. Find the equation of the locus S_2 of the point M for variable t and show that S_2 is a parabola.

Sketch, on the same diagram, the parabolas S_1 and S_2, and write down the coordinates of their foci.

15 The parameters of the three points A_1, A_2 and A_3 on the parabola are t_1, t_2 and t_3 respectively. The tangents at A_1 and A_2 meet the tangent at A_3 at Q and P respectively; the tangents at A_1 and A_2 meet at R. Given that P is the mid-point of QA_3, prove that

$$2t_2 = t_1 + t_3.$$

Prove also that RP is parallel to A_1A_3.

16 Given that the normal at P $(ap^2, 2ap)$ to the parabola meets the curve again at Q $(aq^2, 2aq)$, prove that $p^2 + pq + 2 = 0$.

Prove that an equation of the locus of the point of intersection of the tangents to the parabola at P and Q is

$$y^2(x + 2a) + 4a^3 = 0.$$

17 A point P on the parabola $(x - a)^2 = 4ay$ has coordinates $x = a + 2at$, $y = at^2$. Find equations of the tangent and the normal to the parabola at P.

Given that the tangent and normal cut Ox at the points T and N respectively, prove that

$$PT^2/TN = at.$$

2.3 The ellipse

An ellipse has two axes of symmetry. Referred to these as coordinate axes, its equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a and b are positive constants. The *major* and *minor* axes AA' and BB' of the ellipse are of length $2a$ and $2b$ respectively (see Fig. 2.6), and the curve has parametric equations for P, $x = a\cos\theta$, $y = b\sin\theta$. The angle θ is called the *eccentric angle* of P.

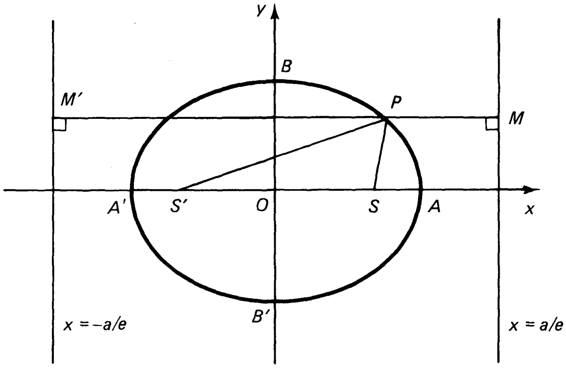

Fig. 2.6 The ellipse $x^2/a^2 + y^2/b^2 = 1$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\bigg/\frac{dx}{d\theta} = -\frac{b}{a}\cot\theta.$$

Hence the tangent at θ is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1.$$

(Check: When $x = a\cos\theta$, $y = b\sin\theta$, the left-hand side is $\cos^2\theta + \sin^2\theta = 1$.) The ellipse has a focus $S(ae, 0)$, where $e < 1$, and corresponding directrix $x = a/e$; the symmetry of the curve implies that there is a second focus $S'(-ae, 0)$ and corresponding directrix $x = -a/e$. The value of b is defined in terms of a and e by the formula

$$b^2 = a^2(1 - e^2).$$

If P is a point on the ellipse, $SP = ePM$, $S'P = ePM'$

$$\Rightarrow \quad SP + S'P = e(2a/e) = 2a.$$

Hence the sum of the focal distances of any point on the ellipse is constant, an important property that can be used as a definition of an ellipse.

Example 6 Prove that the mid-points of a set of parallel chords of an ellipse lie on a fixed straight line through the centre of the ellipse (a *diameter*).

For the mid-point of the chord joining points with parameters θ, ϕ,

$$x = \tfrac{1}{2}a(\cos\theta + \cos\phi), \quad y = \tfrac{1}{2}b(\sin\theta + \sin\phi).$$

The gradient of the chord is

$$\frac{b(\sin\theta - \sin\phi)}{a(\cos\theta - \cos\phi)} = -\frac{2b\cos\frac{1}{2}(\theta+\phi)\sin\frac{1}{2}(\theta-\phi)}{2a\sin\frac{1}{2}(\theta+\phi)\sin\frac{1}{2}(\theta-\phi)}$$

$$= -\frac{b}{a}\cot\tfrac{1}{2}(\theta+\phi) = \text{constant } m \text{ (say)}.$$

But $$\frac{y}{x} = \frac{b\,2\sin\frac{1}{2}(\theta+\phi)\cos\frac{1}{2}(\theta-\phi)}{a\,2\cos\frac{1}{2}(\theta+\phi)\cos\frac{1}{2}(\theta-\phi)} = \frac{b}{a}\tan\tfrac{1}{2}(\theta+\phi) = -\frac{b^2}{a^2m}.$$

Hence the locus of mid-points is $y = -(b^2/a^2m)x$, a diameter of the ellipse.

Example 7 Prove that the lines $y = mx \pm \sqrt{(a^2m^2+b^2)}$ touch the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ for all values of m.

Prove also that the feet of the perpendiculars from the foci of the ellipse to any tangent lie on the auxiliary circle of the ellipse.

For the intersections of the line $y = mx + c$ and the ellipse we have

$$b^2x^2 + a^2(mx+c)^2 = a^2b^2$$

$$\Leftrightarrow \quad (b^2 + a^2m^2)x^2 + 2a^2cmx + a^2(c^2 - b^2) = 0.$$

The line touches the ellipse if this quadratic has equal roots; that is, if

$$a^4c^2m^2 = (b^2 + a^2m^2)a^2(c^2 - b^2)$$

$$\Leftrightarrow \quad b^2c^2 = b^2(b^2 + a^2m^2)$$

$$\Leftrightarrow \quad c^2 = b^2 + a^2m^2.$$

The perpendicular from the focus $(ae, 0)$ to $y = mx + \sqrt{(a^2m^2+b^2)}$ is

$$my + x - ae = 0.$$

To find the locus of the foot of the perpendicular, we must eliminate m from the equations of the two lines. This gives

$$(y - mx)^2 = a^2m^2 + b^2,$$

$$(my + x)^2 = a^2e^2 = a^2 - b^2.$$

Adding,

$$(1 + m^2)(x^2 + y^2) = a^2(1 + m^2)$$

$$\Leftrightarrow \quad x^2 + y^2 = a^2.$$

This is an equation of the circle on the major axis of the ellipse as diameter, known as the *auxiliary circle* of the ellipse.

The reader should verify that the foot of the perpendicular from the other focus also lies on the auxiliary circle, and should note that we have used the same technique as in Example 11 on p. 27. The use of $y = mx + \sqrt{(a^2m^2+b^2)}$ as a tangent is an exception to the rule that parametric equations should be used; when the latter are trigonometric rather than algebraic these exceptions tend to be more frequent.

Example 8 Show that the line $lx + my = n$ touches the ellipse

$$x^2/a^2 + y^2/b^2 = 1$$

if and only if

$$a^2l^2 + b^2m^2 = n^2. \tag{i}$$

Let the point (x_1, y_1) lie on the ellipse. The tangent to the ellipse at (x_1, y_1) is

$$xx_1/a^2 + yy_1/b^2 = 1. \tag{ii}$$

Suppose this is the line

$$lx + my = n, \tag{iii}$$

that is, we assume this line is a tangent. Then equations (ii) and (iii) must be identical except for a common factor, and so the coefficients of x, y and the constant term must be proportional

$$\Leftrightarrow \quad \frac{x_1/a^2}{l} = \frac{y_1/b^2}{m} = \frac{1}{n}$$

$$\Leftrightarrow \quad x_1 = \frac{la^2}{n}, \quad y_1 = \frac{mb^2}{n}. \tag{iv}$$

But the point (x_1, y_1) lies on the ellipse

$$\Leftrightarrow \quad \frac{x_1{}^2}{a^2} + \frac{y_1{}^2}{b^2} = 1$$

$$\Leftrightarrow \quad l^2a^2 + m^2b^2 = n^2.$$

It follows that if the line is a tangent, then condition (i) must be true; conversely, if condition (i) is true, then the line is a tangent at the point defined by (iv).

Note that here we illustrate an important feature of coordinate geometry wherein both a theorem and its converse are proved simultaneously. To achieve this, it must be logically possible to prefix each step in the proof with a double arrow $\Leftrightarrow$.

Exercise 2.3

In this exercise, unless otherwise stated, references are to the ellipse $x^2/a^2 + y^2/b^2 = 1$, with parametric equations $P \equiv (a\cos\theta, b\sin\theta)$.

1 Show that equations of the tangent and normal to the ellipse at the point $(a\cos\theta, b\sin\theta)$ are respectively

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1,$$

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2.$$

Note: In the remainder of this exercise, the results of Question **1** should be used where appropriate.

2 Prove that the tangent at $P(a\cos\theta, b\sin\theta)$ to the ellipse intersects the tangent at $Q\ (a\cos\theta, a\sin\theta)$ to the circle $x^2 + y^2 = a^2$ on the x-axis.

3 The tangent at a point P of the ellipse meets the parabola $y^2 = 4ax$ at the points Q, R, and is such that the mid-point of QR lies on the line $y = -2a$. Prove that the

product of the lengths of the perpendiculars from $(a, 0)$ and $(-a, 0)$ on to the tangent is $b^2/2$.

4 The normal at P meets the axes at Q and R, and the mid-point of QR is M. Show that the locus of M is an ellipse. Sketch, on the same diagram, the two ellipses and show that they have the same eccentricity.

5 The points M and N are the feet of the perpendiculars from the foci of the ellipse to a variable tangent. Prove that M and N lie on the circle $x^2 + y^2 = a^2$.

Prove that the length of MN is greatest when the tangent is parallel to Ox.

6 The equation of a diameter AB of the ellipse is $y = mx$. Prove that the mid-points of all chords of the ellipse parallel to AB lie on another diameter CD, and find the equation of CD.

Prove that, for all m, $AB^2 + CD^2$ is constant.

7 P is any point on the ellipse and the tangent at P meets the coordinate axes at Q, R. If P is the mid-point of QR, show that P lies on a diagonal of the rectangle which circumscribes the ellipse and has its sides parallel to the axes of coordinates.

8 P_1 and P_2 are two points on the ellipse such that θ has the value θ_1 at P_1 and the value θ_2 at P_2. Given that the tangents to the ellipse at P_1 and P_2 meet on the line $ay = bx$, prove that

$$\theta_1 + \theta_2 = \pi/2 \quad \text{or} \quad 5\pi/2.$$

9 The tangent to the ellipse at P and the tangent at $Q(-a\sin\theta, b\cos\theta)$ meet at the point T. Show that $OPTQ$ is a parallelogram and find its area.

Show also that, as θ varies, the locus of the point T is the ellipse

$$x^2/a^2 + y^2/b^2 = 2.$$

10 A, B are the points $(a\cos\theta, b\sin\theta)$, $(a\cos\phi, b\sin\phi)$ on the ellipse. Given that (x_1, y_1) is the mid-point of AB, prove that

(i) $ay_1 = bx_1 \tan\dfrac{\theta+\phi}{2}$, (ii) $b^2{x_1}^2 + a^2{y_1}^2 = a^2b^2\cos^2\dfrac{\theta-\phi}{2}$.

Given also that AB subtends a right-angle at the origin, show that

$$(a^2 + b^2)\cos(\theta - \phi) + (a^2 - b^2)\cos(\theta + \phi) = 0.$$

Hence prove that the locus of mid-points of chords of the ellipse which subtend a right-angle at the origin is

$$(a^2 + b^2)(b^2x^2 + a^2y^2)^2 = a^2b^2(b^4x^2 + a^4y^2).$$

11 The point P $(a\cos\theta, b\sin\theta)$ on the ellipse corresponds to the point $A(a\cos\theta, a\sin\theta)$ on the auxiliary circle $x^2 + y^2 = a^2$, and OP meets the auxiliary circle at $B(a\cos\phi, a\sin\phi)$. Prove that the tangent to the ellipse at the point Q $(a\cos\phi, b\sin\phi)$, corresponding to B, is perpendicular to OA.

Prove also that, if this tangent meets OA at T, then $OP = OT$.

12 The tangent at a point P on the ellipse meets a directrix at Q. If S is the corresponding focus, prove that $\angle PSQ$ is a right angle.

Given that PS meets the ellipse again in P', prove that $P'Q$ is a tangent to the ellipse.

13 Prove that the locus of intersection of perpendicular tangents to the ellipse is the circle $x^2 + y^2 = a^2 + b^2$, the *director circle* of the ellipse.

14 The line $y = mx + c$ cuts the ellipse $x^2 + 4y^2 = 16$ in the points P and Q. Show that the coordinates of G, the mid-point of PQ, are $[-4mc/(4m^2 + 1), c/(4m^2 + 1)]$.

If the chord PQ passes through the point $(2, 0)$, show that G lies on the ellipse $x^2 + 4y^2 = 2x$.

15 The normal at a variable point P on an ellipse of eccentricity e meets the axes of the ellipse in Q, R. Prove that the locus of the mid-point of QR is an ellipse also of eccentricity e.

2.4 The hyperbola

A hyperbola has two axes of symmetry, and its equation with respect to these as coordinate axes is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

with parametric equations $x = a\sec\theta$, $y = b\tan\theta$. (Check: When $x = a\sec\theta$, $y = b\tan\theta$, the left-hand side is $\sec^2\theta - \tan^2\theta = 1$.)

The hyperbola has two distinct branches (see Fig. 2.7) which approach the lines $y = \pm bx/a$; these lines are asymptotes to the curve. As for the ellipse, there are two foci $(\pm ae, 0)$ and two directrices $x = \pm a/e$, with $b^2 = a^2(e^2 - 1)$ where $e > 1$. Also

$$|SP - S'P| = 2a.$$

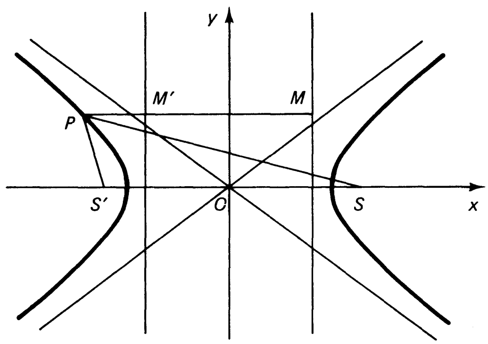

Fig. 2.7 The hyperbola $x^2/a^2 - y^2/b^2 = 1$

Example 9 Write down equations of the two asymptotes of the hyperbola $x^2/9 - y^2/16 = 1$.

The tangent to the hyperbola at the point $P(3\sec\theta, 4\tan\theta)$ meets the asymptotes at X and Y. Show that

(a) P is the mid-point of XY,

(b) the area of the $\triangle XOY$ is independent of θ.

The equations of the asymptotes are $y = \pm 4x/3$.

The tangent at $P \equiv \frac{x}{3}\sec\theta - \frac{y}{4}\tan\theta = 1$.

(a) This tangent meets the asymptotes where $\frac{x}{3}(\sec\theta \pm \tan\theta) = 1$

$\Rightarrow$ x-coordinate of mid-point of XY is

$$\frac{3}{2}\left(\frac{1}{\sec\theta - \tan\theta} + \frac{1}{\sec\theta + \tan\theta}\right) = 3\sec\theta,$$

y-coordinate of mid-point of XY is

$$\frac{4}{2}\left(\frac{1}{\sec\theta - \tan\theta} - \frac{1}{\sec\theta + \tan\theta}\right) = 4\tan\theta$$

$$\Leftrightarrow \quad P \text{ is the mid-point of } XY.$$

(b) If $X \equiv (x_1, y_1)$,

$$OX^2 = x_1{}^2(1 + \tfrac{16}{9}) = 25x_1{}^2/9$$

$$\Rightarrow \quad OX = \tfrac{5}{3}x_1 = 5/(\sec\theta - \tan\theta).$$

Similarly, $OY = 5/(\sec\theta + \tan\theta) \quad \Rightarrow \quad OX.OY = 25$ (since $\sec^2\theta - \tan^2\theta = 1$). $\triangle XOY = \frac{1}{2}OX.OY \sin X\hat{O}Y$, and hence is independent of θ.

2.5 The rectangular hyperbola

If $b = a$, then $e = \sqrt{2}$ and the equation of the hyperbola becomes $x^2 - y^2 = a^2$, with asymptotes at right angles. If the curve is referred to these asymptotes as coordinate axes (Fig. 2.8), its equation becomes $xy = c^2$, $c > 0$, and we can then use $(ct, c/t)$ as parametric coordinates. Then $\dfrac{dy}{dx} = -\dfrac{1}{t^2}$, and the tangent at t

$$\equiv y - \frac{c}{t} = -\frac{1}{t^2}(x - ct) \equiv t^2y + x = 2ct. \tag{2.3}$$

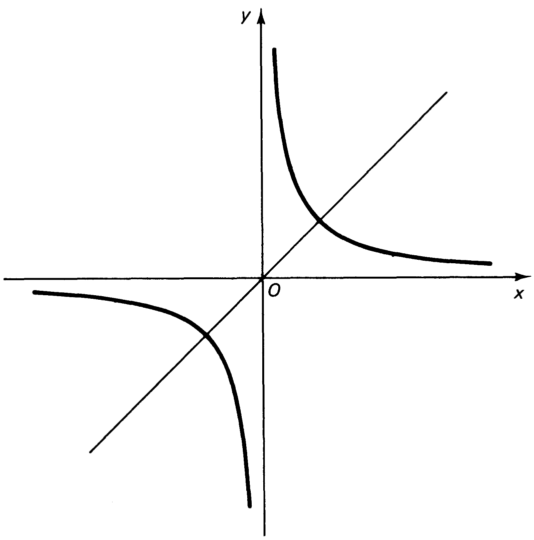

Fig. 2.8 The rectangular hyperbola $xy = c^2$, $c > 0$

Example 10 If three points P_1, P_2 and P_3 lie on $xy = c^2$, prove that the orthocentre of $\triangle P_1P_2P_3$ also lies on the curve.

The line through P_2 perpendicular to P_1P_3

$$\equiv y - \frac{c}{t_2} = t_1t_3(x - ct_2).$$

The line through P_3 perpendicular to P_1P_2

$$\equiv y - \frac{c}{t_3} = t_1t_2(x - ct_3).$$

These lines meet where

$$c\left(\frac{1}{t_3} - \frac{1}{t_2}\right) = t_1x(t_3 - t_2) \Rightarrow x = -\frac{c}{t_1t_2t_3}.$$

By substitution, $y = -ct_1t_2t_3$. Hence the orthocentre, the meet of the altitudes of $\triangle P_1P_2P_3$, lies on the curve $xy = c^2$. [It is the point with parameter $-1/(t_1t_2t_3)$.]

Example 11 Find the locus of the foot of the perpendicular from the origin to a variable tangent of a rectangular hyperbola.

The tangent at A to $xy = c^2 \equiv t^2y + x = 2ct$. The perpendicular from the origin to this tangent $\equiv y - t^2x = 0$. Substituting for t from the second equation,

$$(y/x)y + x = 2c\sqrt{(y/x)}$$

$$\Rightarrow \quad x^2 + y^2 = 2c\sqrt{(xy)} \quad \Rightarrow \quad (x^2 + y^2)^2 = 4c^2xy.$$

This is the equation of the required locus.
Note: Since the coordinates (x, y) of any point of the locus satisfy both equations, we only need to eliminate t between the two equations, rather than solving the equations for x and y in terms of t, and eliminating t. *This technique is often useful in locus problems.*

Example 12 A circle intersects the hyperbola $x = ct$, $y = c/t$ in the points P, Q, R, S. The mid-point of PQ is at the origin. Show that the mid-point of RS is at the centre of the circle.

Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + d = 0$. Then the roots of the equation for t,

$$c^2t^2 + \frac{c^2}{t^2} + 2gct + 2f\frac{c}{t} + d = 0,$$

are the parameters t_1, t_2, t_3, t_4 of P, Q, R, S respectively. Multiplying by t^2, we obtain the quartic

$$c^2t^4 + 2gct^3 + dt^2 + 2fct + c^2 = 0.$$

Since O is the mid-point of PQ, $t_1 + t_2 = 0$. The sum of the roots of the quartic

gives $t_1 + t_2 + t_3 + t_4 = -2g/c$

$$\Rightarrow \quad t_3 + t_4 = -2g/c \quad \Leftrightarrow \quad g = -c(t_3 + t_4)/2.$$

Also

$$t_2t_3t_4 + t_3t_4t_1 + t_4t_1t_2 + t_1t_2t_3 = -2f/c = t_3t_4(t_1 + t_2) + t_1t_2(t_3 + t_4)$$

$$\Rightarrow \quad -f = ct_1t_2(t_3 + t_4)/2 \quad \text{since } t_1 + t_2 = 0.$$

The product of the roots of the quartic gives $t_1t_2t_3t_4 = 1 \Rightarrow t_1t_2 = 1/(t_3t_4)$

$$\Rightarrow \quad -f = c(t_3 + t_4)/(2t_3t_4) = c\left(\frac{1}{t_3} + \frac{1}{t_4}\right)\Big/2.$$

Hence the centre $(-g, -f)$ of the circle is the mid-point of RS.

Exercise 2.5

1 Show that equations of the tangent and normal to the hyperbola

$$x^2/a^2 - y^2/b^2 = 1$$

at the point P $(a \sec\theta, b \tan\theta)$ are respectively

$$\frac{x \sec\theta}{a} - \frac{y \tan\theta}{b} = 1,$$

$$ax \tan\theta + by \sec\theta = (a^2 + b^2) \sec\theta \tan\theta.$$

In Questions **2–8** of this exercise references are to the hyperbola of Question **1**, and the equations of the tangent and normal given there should be used where required.

2 The ordinate at X meets an asymptote at Y. The tangent at X meets the same asymptote at Z. The normal at X meets the x-axis at W. Prove that the angle ZYW is a right angle.

3 Show that the product of the areas of the two triangles formed by the tangent and normal at P and the coordinate axes is independent of θ.

Find the locus of the circumcentre of the triangle formed by the tangent and the coordinate axes.

4 Find the coordinates of the points A, B in which the tangent at P meets the asymptotes of a hyperbola.

Prove that $OA \,.\, OB = a^2 + b^2$, and obtain an expression for the area of triangle OAB.

5 The normal meets Ox at Q and Oy at R. Show that both the ratio of the x-coordinate of P to that of Q and the ratio of the y-coordinate of P to that of R are independent of the position of P.

Show also that, as P varies, the locus of the mid-point of QR is a hyperbola, and that each asymptote of this hyperbola is perpendicular to an asymptote of the given hyperbola.

6 Show that the locus of the foot of the perpendicular from the origin to a variable tangent to the hyperbola is the circle

$$x^2 + y^2 = a^2 - b^2.$$

[This is a real circle when $a > b$ and is called the *director circle* of the hyperbola.]

7 The tangents to the hyperbola at the points A and B on the curve meet at T. Given that M is the mid-point of AB, prove that TM passes through the centre of the hyperbola. Prove also that the product of the gradients of AB and TM is constant.

8 Show that the straight line

$$x\cos\alpha + y\sin\alpha = p$$

is a tangent to the hyperbola if

$$a^2\cos^2\alpha - b^2\sin^2\alpha = p^2,$$

and find the coordinates of the point of contact.

Obtain equations of those tangents to the hyperbola $9x^2 - 16y^2 = 144$ which touch the circle $x^2 + y^2 = 9$.

9 Find the angle between the asymptotes of the hyperbola

$$x^2 - \frac{y^2}{3} = 1.$$

10 Prove that the ellipse $4x^2 + 9y^2 = 36$ and the hyperbola $4x^2 - y^2 = 4$ have the same foci, and that they intersect at right angles.

Find an equation of the circle through the points of intersection of the two conics.

11 Prove that an equation of the chord joining the points $R\left(cr, \frac{c}{r}\right)$ and $S\left(cs, \frac{c}{s}\right)$ on the rectangular hyperbola $xy = c^2$ is

$$x + rsy = c(r + s).$$

Deduce that the equations of the tangent and normal to the curve at the point $P \equiv (ct, c/t)$ are respectively

$$x + t^2y = 2ct,$$

$$y = t^2x + \frac{c}{t} - ct^3.$$

In Questions **12**–**23** of this exercise references are to the rectangular hyperbola of Question **11** and the notation and results given there should be used where appropriate.

12 Given that this chord passes through the point $(cr + cs - c, c)$, show that the tangents at R and S meet on the line $y = x$.

13 Find equations of the tangents to the rectangular hyperbola which are parallel to the line $y + 4x = 0$. Find also the perpendicular distance between these tangents.

14 If the normal at P meets the line $y = x$ at N, and O is the origin, show that, provided that $t \neq 1$, $OP = PN$.

The tangent to the hyperbola at P meets the line $y = x$ at T. Prove that $OT.ON = 4c^2$.

15 Three points P_1, P_2, P_3 lie on the rectangular hyperbola. Prove that

(a) if P_1P_2 and P_1P_3 are equally inclined to the axes of coordinates, then P_2P_3 passes through the origin O,

(b) if angle $P_2P_1P_3$ is a right angle, then P_2P_3 perpendicular to the tangent at P_1.

16 The points $P_1(ct_1, c/t_1)$ and $P_2(ct_2, c/t_2)$ vary so that P_1P_2 touches the circle $x^2 + y^2 = c^2$. Prove that the point of intersection of the tangents to the hyperbola at P_1 and P_2 lies on another fixed circle.

17 The normal at the point P on $xy = c^2$ meets the hyperbola $x^2 - y^2 = a^2$ at Q and R. Prove that P is the mid-point of QR.

18 Show that the normal to the rectangular hyperbola at P meets the curve again at $Q(-c/t^3, -ct^3)$.

The tangent to the hyperbola at P meets the coordinate axes at A, B. Show that the area of the triangle QAB is proportional to $(t^2 + t^{-2})^2$.

Deduce that as t varies the least area of the triangle is $4c^2$.

19 Four distinct points P_1, P_2, P_3, P_4 lying on a rectangular hyperbola have parameters t_1, t_2, t_3, t_4 respectively.

(a) Find the gradient of the chord P_1P_2.

(b) Show that, if P_1P_2 is parallel to P_4P_3 and P_1P_4 is parallel to P_2P_3, then $t_2 = -t_4$. Deduce that, if a parallelogram is inscribed in a rectangular hyperbola, then its centre is at the centre of the hyperbola.

(c) Show that if $P_1P_2P_3P_4$ is a rectangle, then $t_1t_2 = \pm 1$. Deduce that it is impossible to inscribe a square in a rectangular hyperbola.

20 Given that the tangents from the point $R\,(h, k)$ to the rectangular hyperbola meet the hyperbola at P and Q, show that the parameters of the points P and Q are the roots of the quadratic equation

$$kt^2 - 2ct + h = 0.$$

Prove that the mid-point of PQ is (c^2k^{-1}, c^2h^{-1}).

Deduce that, if R lies on the line l with equation

$$ax + by = 0,$$

then the mid-point of PQ also lies on l.

21 X and Y are two points on a rectangular hyperbola. The tangents at X and Y to the hyperbola meet one of its asymptotes at A and C respectively, and meet the other asymptote at B and D respectively. Prove that AD and BC are both parallel to XY and are equidistant from it.

22 The gradient m of the chord PQ of the rectangular hyperbola is constant and positive. Show that for all positions of PQ, there are two fixed points through which the circle on PQ as diameter passes.

Show also that, if the chord RS is perpendicular to PQ, the circle on RS as diameter cuts orthogonally the circle on PQ as diameter.

23 A straight line passes through the fixed point $A\,(3, 2)$ and meets the coordinate axes at P and Q. If the mid-point of PQ is X, show that, as the gradient of the line varies, the point X moves on the curve whose equation is

$$2xy = 2x + 3y.$$

Verify that A lies on this curve. Show also that, if the tangent at A to this curve meets the coordinate axes at H and K, then A is the mid-point of HK.

2.6 The line-pair through the origin

If $(x - 2y)(5x + y) = 0$, the point (x, y) lies either on the line $x - 2y = 0$ or the line $5x + y = 0$. The given equation therefore represents a pair of straight lines through the origin.

Example 13 Interpret the equation $7x^2 + 19xy - 6y^2 = 0$.

The equation can be written $(x + 3y)(7x - 2y) = 0$. Hence it represents the line-pair $x + 3y = 0$, $7x - 2y = 0$ through the origin.

The equation $ax^2 + 2hxy + by^2 = 0$ can be written

$$a\left(\frac{x}{y}\right)^2 + 2h\left(\frac{x}{y}\right) + b = 0, \ (y \neq 0).$$

This quadratic in x/y has real roots if and only if $h^2 \geqslant ab$. The left-hand side of the equation has real distinct factors, say $(p_1x + q_1y)(p_2x + q_2y)$, if $h^2 > ab$, and hence, if this condition is satisfied, the equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of (distinct) straight lines through the origin.

If $h^2 = ab$, the two lines coincide; and if $h^2 < ab$ the origin is the only point with coordinates satisfying the equation.

Example 14 Find the condition for $ax^2 + 2hxy + by^2 = 0$ to represent a pair of perpendicular straight lines.

If $ax^2 + 2hxy + by^2 \equiv (p_1x + q_1y)(p_2x + q_2y)$, the condition for perpendicularity is

$$\frac{p_1}{q_1}.\frac{p_2}{q_2} = -1 \quad \Leftrightarrow \quad p_1p_2 + q_1q_2 = 0 \quad \Leftrightarrow \quad a + b = 0.$$

This is the required condition. Note that

$$a + b = 0 \quad \Rightarrow \quad ab < 0 \quad \Rightarrow \quad h^2 > ab.$$

Example 15 Find an equation of the angle bisectors of the line-pair

$$ax^2 + 2hxy + by^2 = 0.$$

Any point on either angle bisector is equidistant from the lines $p_1x + q_1y = 0$ and $p_2x + q_2y = 0$.
Hence, using (1.6), the equation of the angle bisectors is

$$\left[\frac{p_1x + q_1y}{\sqrt{(p_1{}^2 + q_1{}^2)}} - \frac{p_2x + q_2y}{\sqrt{(p_2{}^2 + q_2{}^2)}}\right]\left[\frac{p_1x + q_1y}{\sqrt{(p_1{}^2 + q_1{}^2)}} + \frac{p_2x + q_2y}{\sqrt{(p_2{}^2 + q_2{}^2)}}\right] = 0$$

$$\equiv \frac{(p_1x + q_1y)^2}{p_1{}^2 + q_1{}^2} - \frac{(p_2x + q_2y)^2}{p_2{}^2 + q_2{}^2} = 0$$

$$\equiv (x^2 - y^2)(p_1{}^2q_2{}^2 - p_2{}^2q_1{}^2) - 2xy(p_1p_2 - q_1q_2)(p_1q_2 - p_2q_1) = 0$$

$$\div \ (p_1q_2 - p_2q_1) \equiv (x^2 - y^2)(p_1q_2 + p_2q_1) - 2xy(p_1p_2 - q_1q_2) = 0.$$

But

$$ax^2 + 2hxy + by^2 \equiv (p_1x + q_1y)(p_2x + q_2y)$$

$$\Rightarrow \quad p_1p_2 = a, \quad q_1q_2 = b, \quad p_1q_2 + p_2q_1 = 2h.$$

Hence an equation of the angle bisectors is

$$h(x^2 - y^2) - (a - b)xy = 0.$$

Example 16 Find an equation of the pair of straight lines joining the origin to the intersections of the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the line $lx + my = n$.

The points of intersection satisfy the equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$$(lx + my)^2 = n^2.$$

Cross-multiplying to give a homogeneous equation of degree 2 in x and y, we obtain

$$n^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = (lx + my)^2$$

$$\Rightarrow \quad \left(l^2 - \frac{n^2}{a^2}\right)x^2 + 2lmxy + \left(m^2 - \frac{n^2}{b^2}\right)y^2 = 0.$$

This is the required line-pair.

Exercise 2.6

1 Show that, as θ varies, the line-pair represented by the equation

$$(x^2 + y^2)\cos 2\theta + 2xy = 0$$

is always real, and find the angle between the lines.

2 (a) Find separate equations of the lines

$$3x^2 - 8xy - 3y^2 = 0.$$

Sketch the two lines.

(b) One of the lines given by the equation

$$px^2 + 2qxy + ry^2 = 0$$

passes through the point $(1, 2)$ and the other passes through the point $(-3, 4)$. Find the values of $p : q : r$.

(c) The line $lx + my = 1$ meets the parabola $y^2 = 4ax$ at the points A, B. Write down an equation of the pair of lines joining the origin O to A and B.

Hence show that all chords of the parabola which subtend a right angle at O pass through the point $(4a, 0)$.

3 Prove that the equation $3x^2 - 2y^2 - 5xy = 0$ represents a pair of straight lines, and that the tangent of one of the angles between the line-pair is 7.

Find an equation of the angle bisectors of this line-pair.

4 Prove by translating the origin, or otherwise, that the equation

$$7x^2 + 16xy - 8y^2 - 14x - 16y + 7 = 0$$

represents a pair of straight lines through the point $(1, 0)$.

Prove also that the angle bisectors of this pair form a rectangle with the angle bisectors of the pair of lines with equation

$$12x^2 + 16xy - 3y^2 = 0.$$

5 Prove that the equation

$$8x^2 + 38xy - 33y^2 + 26x + 68y + 21 = 0$$

represents a pair of straight lines.

Show also that the tangent of the angle between these lines is 2.

6 Show that

$$x^2 + 4xy - 2y^2 + 6x - 12y - 15 = 0$$

represents a pair of straight lines, and that these lines together with the pair of lines $x^2 + 4xy - 2y^2 = 0$ form a rhombus.

Miscellaneous exercise 2

1 The tangents at points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ on the parabola $y^2 = 4ax$ intersect at T. Find the coordinates of T.

Given that PQ is of constant length d, show that T lies on the curve whose equation is

$$(y^2 - 4ax)(y^2 + 4a^2) = a^2d^2.$$

2 Prove that the line $lx + my + n = 0$ touches the parabola $y^2 = 4ax$ if $am^2 = ln$.

State the condition that the same line should touch the parabola $x^2 = 4by$. Deduce that an equation of the common tangent to the two parabolas is

$$a^{1/3}x + b^{1/3}y + (ab)^{2/3} = 0.$$

Prove also that the common tangent divides the common chord externally in the ratio 1 : 9.

3 C is the mid-point of a variable chord AB of the parabola $y^2 = 4ax$. The tangents at A and B to the parabola meet at G. Prove (a) that GC is parallel to the axis of the parabola and (b) that, if AB subtends a right angle at the vertex, then the locus of C is

$$2ax = y^2 + 8a^2.$$

4 The parabolas $x^2 = 4ay$ and $y^2 = 4ax$ meet at the origin and at the point P. The tangent to $x^2 = 4ay$ at P meets $y^2 = 4ax$ again at A, and the tangent to $y^2 = 4ax$ at P meets $x^2 = 4ay$ again at B. Find the tangent of $A\hat{P}B$ and prove that AB is a common tangent to the two parabolas.

5 From the point (h, k) tangents are drawn to the parabola $y^2 = 4ax$. Show that the area of the triangle formed by the tangents and the chord of contact is $(k^2 - 4ah)^{3/2}/2a$.

6 A variable chord joining points P_1 and P_2 on the parabola $y^2 = 4ax$ passes through the fixed point $A(2a, 0)$. Prove that the tangents to the parabola at P_1 and P_2 meet on a fixed straight line, and that the normals at P_1 and P_2 meet on a fixed parabola.

7 The points P, Q, R on the parabola $y^2 = 4ax$ have coordinates $(ap^2, 2ap)$, $(aq^2, 2aq)$, $(ar^2, 2ar)$ respectively.

(a) If RP is perpendicular to RQ, prove that $r^2 + (p + q)r + pq + 4 = 0$.

(b) If PQ passes through the focus, prove that $pq + 1 = 0$.

Deduce that, if the circle on a focal chord as diameter cuts the parabola again in real points, the chord is inclined to the axis at an angle not exceeding $\pi/6$.

8 The tangent and the normal to the parabola $y^2 = 4ax$ at P meet the x-axis at T and G respectively, and M is the foot of the perpendicular from P to the line $x = -a$; S is the point $(a, 0)$. Show that $ST = PM = SP$ and deduce that PT bisects the angle SPM. Show also that PG bisects the angle between PS and MP produced.

9 P and Q are two points on the parabola $y^2 = 4ax$ such that the tangents at P and Q to the parabola make equal angles with the fixed straight line $y = kx$. Prove that, as P and Q vary, PQ passes through a fixed point on the directrix of the parabola.

Prove that the mid-points of the chords PQ lie on the curve

$$ky^2 - 2akx + a(k^2 - 1)y - 2a^2k = 0.$$

10 The tangents to the parabola $y^2 = 4ax$ at the points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ meet at R. Find the coordinates of the point R, and show that the area of the triangle PQR is

$$\left|\tfrac{1}{2}a^2(p - q)^3\right|.$$

Given that the points P and Q move so that the area of the triangle PQR is $4a^2$, find an equation of the locus of R.

11 The normals at three points P_1, P_2, P_3 of the parabola $y^2 = 4ax$ meet at a point N. Prove that the centroid of the triangle $P_1P_2P_3$ lies on the axis of the parabola.

If N coincides with P_1, prove that P_2P_3 passes through a fixed point.

12 A chord of the parabola $y^2 = 4ax$ subtends a right angle at a fixed point P of the parabola. Prove that all such chords pass through a fixed point on the normal at P to the parabola (called the *Frégier point* of P).

Prove that the Frégier points of all points on $y^2 = 4ax$ lie on an equal parabola, and that the chords of $y^2 = 4ax$ whose mid-points are these Frégier points touch this equal parabola.

13 Prove that $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$ is an equation of the circle on the line joining the points (x_1, y_1) and (x_2, y_2) as diameter.

P and Q are the ends of a focal chord of the parabola $y^2 = 4ax$. Prove that the circle on PQ as diameter touches the directrix of the parabola. Prove also that if the circle meets the parabola in two real points with parameters t_1, t_2, in addition to P and Q, then $t_1t_2 = 3$.

14 (a) The normal to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the point $(a\cos\phi, b\sin\phi)$ meets the x-axis at Q. Prove that the distance of Q from the centre of the ellipse does not exceed $(a^2 - b^2)/a$.

(b) A ladder 12 m long slides in a vertical plane with its ends in contact with a vertical wall and a horizontal floor. Find the locus of a point on the ladder 4 m from its foot.

15 Prove that the tangent to an ellipse at any point bisects one pair of angles between the lines joining that point to the foci of the ellipse.

An ellipse has semi-axes of length $2a$, $a\sqrt{3}$. The distance of a point P on this ellipse from one focus S is $\tfrac{3}{2}a$, and the tangent at P to the ellipse cuts the major axis at T. Calculate the length of ST.

16 P and Q are any two points on the ellipse $x^2/a^2 + y^2/b^2 = 1$ with coordinates $(a\cos\theta, b\sin\theta)$ and $(a\cos\varphi, b\sin\varphi)$. Prove that the chord PQ has equation

$$\frac{x}{a}\cos\frac{1}{2}(\theta + \varphi) + \frac{y}{b}\sin\frac{1}{2}(\theta + \varphi) = \cos\frac{1}{2}(\theta - \varphi).$$

Given that PQ subtends a right angle at the point $A(a, 0)$, show that

$$\tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\varphi = -b^2/a^2.$$

Deduce that PQ passes through a fixed point on the x-axis.

17 Prove that the mid-points of a system of chords of the ellipse $x^2/a^2 + y^2/b^2 = 1$ parallel to a diameter d_1 lie on a diameter d_2 (said to be *conjugate* to d_1), and that d_1 is conjugate to d_2.

Given that the diameters through points on the ellipse with eccentric angles θ_1 and θ_2 are conjugate, prove that θ_1 and θ_2 differ by a right angle.

Given that Q is one end of the diameter conjugate to a diameter P_1P_2, prove that the diameters parallel to P_1Q and P_2Q are conjugate.

18 The tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the point $P(a\cos\theta, b\sin\theta)$ meets the tangents at the ends of the major axis in the points Q, R. Prove that the circle on QR as diameter cuts Ox at two points whose positions are independent of θ.

19 A variable straight line $lx + my = 1$ cuts the hyperbola $x^2/a^2 - y^2/b^2 = 1$ at P and Q. Find an equation of the pair of lines OP, OQ.

Given that $b > a$ and OP and OQ are perpendicular, prove that the line PQ touches a fixed circle whose centre is O. Give an equation of the circle.

20 The foot of the perpendicular from the origin O to the tangent at the point $P(ct, c/t)$ on the hyperbola $xy = c^2$ is Q.

(a) Prove that $OP\,.\,OQ$ is constant.

(b) Find an equation of the locus of Q, and sketch this locus.

21 Find an equation of the normal to the rectangular hyperbola $xy = c^2$ at the point $(ct, c/t)$.

The normals to the rectangular hyperbola at the points P_1, P_2, P_3, P_4 meet at a common point (h, k). Show that the parameters of the points P_1, P_2, P_3, P_4 are roots of the equation

$$ct^4 - ht^3 + kt - c = 0.$$

Show also that the line joining any two of the points P_1, P_2, P_3, P_4 is perpendicular to the line joining the other two.

22 Given that $a > k > b > 0$, prove that the ellipse

$$x^2/a^2 + y^2/b^2 = 1$$

cuts the hyperbola

$$\frac{x^2}{(a^2 - k^2)} + \frac{y^2}{(b^2 - k^2)} = 1$$

in four real points.

If P is any one of these points, prove that the tangents at P to the two curves are perpendicular.

Find an equation of the circle which passes through the four points.

23 Tangents are drawn to the rectangular hyperbola $xy = c^2$ at the points $P(ct_1, c/t_1)$ and $Q(ct_2, c/t_2)$. Find the coordinates of the orthocentre of the triangle formed by these tangents and the line $y = 0$. Given that P and Q vary so that the line PQ passes through a fixed point on the line $y = 0$, show that the orthocentre lies on a fixed parabola.

24 Given that the normal to the hyperbola $xy = c^2$ at the point $x = ct$, $y = c/t$ passes through the point $P(h, k)$, show that

$$ct^4 - ht^3 + kt - c = 0.$$

The normals at four points on the hyperbola meet at P. Show that the sum of the x-coordinates of the four points is h, and that the sum of their y-coordinates is k.

25 S is the parabola with parametric equations $x = at^2$, $y = 2at$, and P_1 and P_2 are the points of S with parameters t_1 and t_2 respectively. Find an equation of the line P_1P_2 and an equation of the circle on P_1P_2 as diameter. Given that this circle cuts S again at P_3 and P_4, show that an equation of P_3P_4 is

$$2x + (t_1 + t_2)y + 2a(t_1t_2 + 4) = 0.$$

26 The chord AB joining points $A(ct_1, c/t_1)$ and $B(ct_2, c/t_2)$ on the rectangular hyperbola $xy = c^2$ is of constant length a. Show that, as the position of the chord varies, the centroid G of the triangle AOB, where O is the origin, moves on the curve

$$(9xy - 4c^2)(x^2 + y^2) = a^2xy.$$

27 The hyperbola $x = ct$, $y = c/t$ is cut in the points P_1, P_2, P_3, P_4 by a circle which has its centre on the y-axis. If the values of t at these points are t_1, t_2, t_3, t_4, show that (a) $t_1t_2t_3t_4 = 1$, (b) $t_1 + t_2 + t_3 + t_4 = 0$.

Given that the equation of the chord P_1P_2 is $x - 3y = 2c$, find an equation of the chord P_3P_4 and an equation of the circle.

3 Polar coordinates, matrices and transformations

3.1 Polar coordinates

The polar coordinates (r, θ) of a point P in a plane provide an alternative, and sometimes convenient, way of describing its position relative to a fixed point, the *pole*, in the plane; for example, '50 km from O, bearing 53·1°' instead of '40 km east and 30 km north from O'. Taking the pole to coincide with the origin of a cartesian system, and the *initial line Ol*, from which θ is measured anti-clockwise, to coincide with the x-axis, Fig. 3.1 enables us to convert from cartesian to polar coordinates thus:

$$x = r\cos\theta, \quad y = r\sin\theta. \tag{3.1}$$

The reverse process is more awkward:

$$r = \sqrt{(x^2 + y^2)}, \quad \theta = \arctan(y/x). \tag{3.2}$$

For example,

$$\text{if } x = y = 1, \quad \text{then } 0 < \theta < \tfrac{1}{2}\pi \quad \Rightarrow \quad \theta = \tfrac{1}{4}\pi, \quad r = \sqrt{2} \quad (P_1);$$

$$\text{if } x = -1, y = -1, \quad \text{then } \pi < \theta < \tfrac{3}{2}\pi \quad \Rightarrow \quad \theta = \tfrac{5}{4}\pi, \quad r = \sqrt{2} \quad (P_2).$$

Note that if x and y are given (and not both zero), then equations (3.1) define just one pair of values (r, θ) for $r > 0$ and $-\pi < \theta \leqslant \pi$ or $0 \leqslant \theta < 2\pi$. To avoid confusion, some writers and examination boards (but not all) adopt this convention ($r > 0$).

However, in practice it is sometimes convenient to allow the alternatives

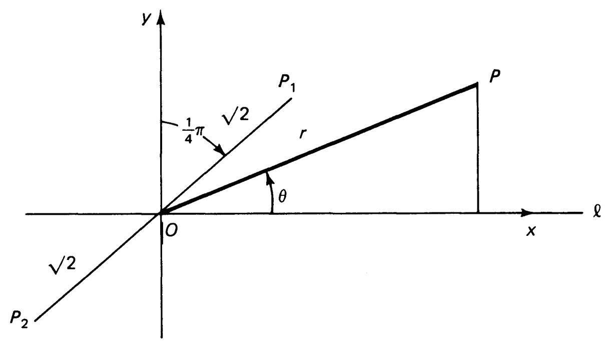

Fig. 3.1

$(+\sqrt{2}, \pi/4)$ and $(-\sqrt{2}, 5\pi/4)$ for P_1, and $(-\sqrt{2}, \pi/4)$ and $(+\sqrt{2}, 5\pi/4)$ for P_2, and in general (r, θ), $(-r, \theta + \pi)$; that is to allow r to have both positive and negative values, positive in the direction OP and negative in the direction PO, but it is never *necessary* to use a negative value for the r-coordinate of a point. In this chapter the convention adopted with regard to values of r will be that negative values of r are to be allowed unless otherwise stated.

3.2 Loci in polar coordinates

(i) The polar equation $\theta = \alpha$ represents a *half-line* through the pole; the other half of the line has equation $\theta = \pi + \alpha (r > 0)$.
(ii) The general straight line $ax + by + c = 0$ has the polar equation $r(a\cos\theta + b\sin\theta) + c = 0$ or $r\cos(\theta - \alpha) = p$.
(iii) The polar equation $r = a$, where $a > 0$, represents the circle with centre the pole and radius a.

If $r = \mathrm{f}(\cos\theta)$, since $\cos(-\theta) = \cos\theta$, the curve is symmetrical about the initial line. If $r = \mathrm{g}(\sin\theta)$, since $\sin(\frac{1}{2}\pi - \theta) = \sin(\frac{1}{2}\pi + \theta)$, the curve is symmetrical about the half-lines $\theta = \pm\frac{1}{2}\pi$. To obtain a curve with polar equation $r = \mathrm{h}(\cos\theta, \sin\theta)$, clearly we need only consider θ in the range $0 \leqslant \theta < 2\pi$ (or $-\pi < \theta \leqslant \pi$). Considerations of symmetry together with some convenient values of θ will usually enable us to sketch a polar curve. However, polar equations will not be used unless they are simple; it may be even simpler to identify a curve from its cartesian equation. For example $r = 2\cos\theta$, $r = 2\sin\theta$, become respectively $x^2 + y^2 = 2x$, $x^2 + y^2 = 2y$ in cartesian coordinates; that is they represent circles with centres $(1, 0)$, $(0, 1)$ respectively, and radius 1.

Example 1 On the curve $r = a\theta$, where $a > 0$, r increases with θ, so that the curve is a *spiral* (Fig. 3.2).

Example 2 The curve $r = a(1 + \cos\theta)$, $a > 0$, must be symmetrical about the initial line; hence we need only consider $0 \leqslant \theta \leqslant \pi$. Clearly, as θ increases from 0 to π, r decreases from $2a$ to 0. The curve is called a *cardioid* (Fig. 3.3).

Example 3 The curve $r = a(1 + 2\cos\theta)$, $a > 0$, is also symmetrical about the initial line. Also $r = 0$ when $\theta = \frac{2}{3}\pi$; and for $\frac{2}{3}\pi < \theta \leqslant \pi$, $r < 0$. The curve, called a *limaçon*, has an outer and inner loop (Fig. 3.4). Had we used the convention $r > 0$, then r is not defined for $-2\pi/3 < \theta \leqslant -\pi$ and there is no inner loop.

Example 4 The lemniscate $r^2 = a^2\cos 2\theta$, $a > 0$, is symmetrical about the initial line and about $\theta = \pm\frac{1}{2}\pi$. Also $r = 0$ when $\theta = \pm\frac{1}{4}\pi$, $\pm\frac{3}{4}\pi$. The curve consists of two loops (Fig. 3.5).

Example 5 If $r = a\sin 3\theta$, $a > 0$, there is symmetry about $\theta = \pm\frac{1}{2}\pi$, since $\sin 3\theta$ can be expressed in terms of $\sin\theta$. Also $r = 0$ when $\theta = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \pi$. The curve consists of *three equal loops* (Fig. 3.6).

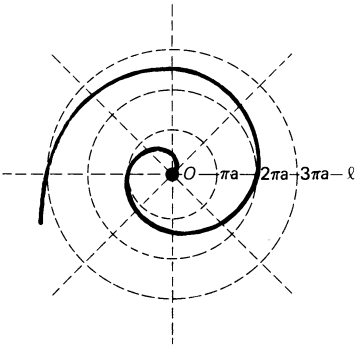

Fig. 3.2 The spiral
$r = a\theta,\ a > 0$

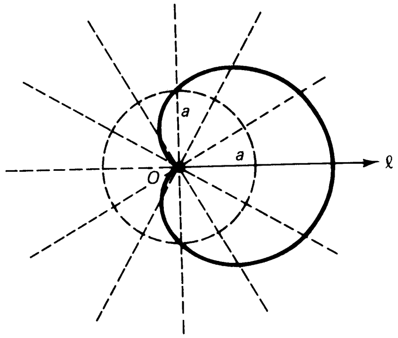

Fig. 3.3 The cardioid
$r = a(1 + \cos\theta),\ a > 0$

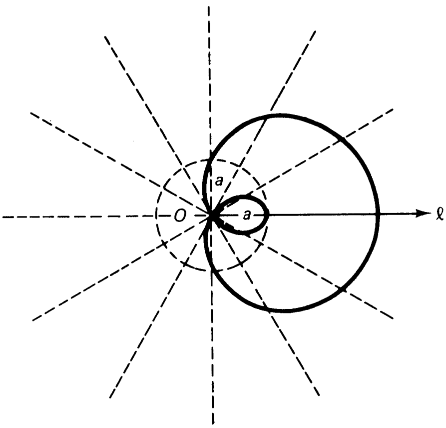

Fig. 3.4 The limaçon $r = a(1 + 2\cos\theta),\ a > 0$

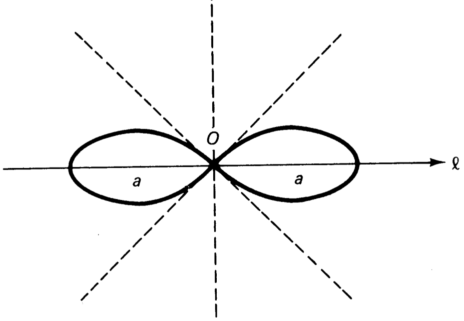

Fig. 3.5 The lemniscate
$r^2 = a\cos 2\theta,\ a > 0$

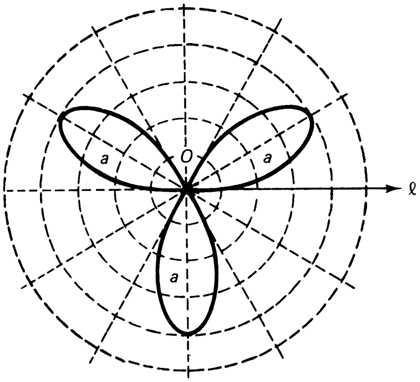

Fig. 3.6 The curve $r = a\sin 3\theta,\ a > 0$

Note that near the pole, where $r = 0$, the curve $f(r, \theta) = 0$ behaves like $f(0, \theta) = 0$, which is an equation whose solutions give specific values of θ defining half-lines through O. In Example 5, near O the equation of the curve approximates to $\sin 3\theta = 0 \quad \Rightarrow \quad \theta = 0, \ \pm\pi/3, \ \pm 2\pi/3, \ \pi$, so that near O the curve looks like six half-lines.

Exercise 3.2

Throughout this exercise take $a > 0$.

In each of Questions **1–7** find, from first principles, a polar equation of the locus defined there and sketch the locus.

1 The straight line parallel to and above the initial line and distant a from the pole.
2 The two straight lines perpendicular to the initial line and distant a from the pole.
3 The straight line joining (a, π) to $(a, \frac{1}{2}\pi)$.
4 The straight line through $(a, 0)$ which makes an angle $\pi/6$ with the initial line.
5 The circle with radius a and centre $(a, 0)$.
6 The circle with radius a and centre (a, π).
7 The parabola with focus at the pole and directrix $r = a \sec \theta$.

In each of Questions **8–11** transform the equation to cartesian form.

8 $r^2 = a^2 \cos 2\theta$.
9 $r = a/(1 + \cos \theta)$.
10 $r = a(1 + 2\cos \theta)$.
11 $r = a \sec(\theta - \alpha)$.

In each of Questions **12–16** transform the equation to polar form.

12 $x^2 + y^2 - 2ax = 0$.
13 $x^2 + y^2 - 2ay = 0$.
14 $(x^2 + y^2)^2 - 2a^2xy = 0$.
15 $x^2 - y^2 = a^2$.
16 $xy = a^2/2$.
17 Sketch, for $0 \leqslant \theta \leqslant \pi/2$, the curve with polar equation $r = 2 \sin 2\theta$.
18 Sketch on the same diagram the curve C_1 with polar equation $r = 2\cos\theta$ and the curve C_2 with polar equation $r = 2$.

Given that P is the point where the curves meet and Q is the point $(2, \pi/2)$ on C_2, show that the centre of the circle OPQ lies on C_1.

19 Sketch the curve $r = 2a(1 + \cos\theta)$.

Find the polar coordinates of the points in which the curve meets the line

$$2r\cos\theta + a = 0.$$

20 Sketch the curve $r = 3 + 2\cos\theta$, marking clearly the polar coordinates of the points on the curve at which r has its greatest and least values.
21 Sketch for $-\pi/4 \leqslant \theta \leqslant \pi/4$ the curve $r = 4\cos 2\theta$.
22 Sketch the curves

$$r = 2\sin\theta, \qquad 0 \leqslant \theta \leqslant \pi,$$
$$r = 2\cos\theta, \quad -\pi/2 \leqslant \theta \leqslant \pi/2,$$

and find the cartesian coordinates of the points of intersection of the curves.

23 In separate diagrams sketch the curves whose equations are
(a) $r = 1 + \theta, 0 \leqslant \theta \leqslant 2\pi$,
(b) $r = 2\sin\theta, 0 \leqslant \theta \leqslant \pi$.

24 Sketch, for $0 \leqslant \theta \leqslant 2\pi$, the curves with polar equations $r = a(1 + \cos\theta)$ and $r = 3a/2$. These curves interest at P and Q. Find the polar coordinates of P and Q, and obtain polar equations of the half-lines OP, OQ and the line PQ.

25 Sketch for $0 \leqslant \theta \leqslant 4\pi$,
(a) the curve $r = k\theta$ (the *spiral of Archimedes*),
(b) the curve $r = k/\theta$ (the *reciprocal spiral*),
(c) the curve $r = e^{k\theta}$ (the *equiangular* or *logarithmic* spiral).

3.3 Area of a sector of a polar curve

If $P(r, \theta)$ and $Q(r + \delta r, \theta + \delta\theta)$ are two adjacent points on a polar curve (Fig. 3.7), the area of the sector enclosed by the arc PQ and the 'radii' OP, OQ, where O in the pole, is approximately equal to that of a sector OPQ' of a circle, where $OP = OQ' = r$ and $P\hat{O}Q' = \delta\theta$; that is $\frac{1}{2}r^2\delta\theta$. Hence the area of a sector of a polar curve between (r_1, θ_1) and (r_2, θ_2) is

$$\lim_{\delta\theta \to 0} \sum \tfrac{1}{2}r^2\,\delta\theta = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta. \tag{3.3}$$

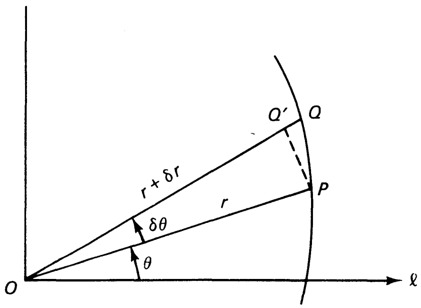

Fig. 3.7

Example 6 Sketch, in the same diagram, the curves $r = 3\cos\theta$, $r = 1 + \cos\theta$, and find the area of the region lying inside both curves.

Expressing $r = 3\cos\theta \equiv r^2 = 3r\cos\theta$ in cartesians, the equation becomes $x^2 + y^2 = 3x$; the circle of centre $(3/2, 0)$ and radius $3/2$. This circle and the cardioid $r = 1 + \cos\theta$ are shown in Fig. 3.8. The curves intersect at P where $\theta = \frac{1}{3}\pi$, $r = 3/2$; the region inside both curves consists of twice the region $OCQP$, made up of the sector OPQ of the cardioid and the minor segment of the circle bounded by the chord OP.
Hence area required

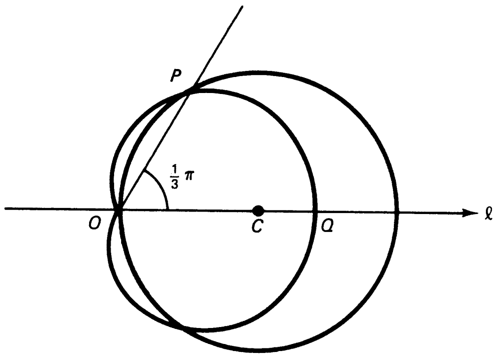

Fig. 3.8 The curves $r = 3\cos\theta$ and $r = 1 + \cos\theta$

$$= 2\left[\frac{1}{2}\left(\frac{3}{2}\right)^2\frac{\pi}{3} - \frac{1}{2}\left(\frac{3}{2}\right)^2\sin\left(\frac{\pi}{3}\right) + \frac{1}{2}\int_0^{\pi/3}(1+\cos\theta)^2\,d\theta\right]$$

$$= \frac{9}{4}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) + \int_0^{\pi/3}[1 + 2\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)]\,d\theta$$

$$= \frac{3}{4}\pi - \frac{9\sqrt{3}}{8} + \left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/3}$$

$$= \frac{3}{4}\pi - \frac{9\sqrt{3}}{8} + \frac{1}{2}\pi + \sqrt{3} + \frac{1}{8}\sqrt{3} = \frac{5}{4}\pi.$$

Exercise 3.3

1 Sketch the curve $r^2 = a^2\cos 2\theta$, where $r \geqslant 0$ and $a > 0$.
(a) Find the angle between the tangents to the curve at the pole.
(b) Find the area of the region enclosed by one loop of the curve.

2 Shade the region C for which

$$r \leqslant 4\cos 2\theta, \quad -\pi/4 \leqslant \theta \leqslant \pi/4.$$

Find the area of C.
State equations of the tangents to the curve $r = 4\cos 2\theta$ at O.

3 Sketch, on the same diagram, the curves $r = 2a\cos 2\theta$, $-\pi/4 \leqslant \theta \leqslant \pi/4$, and $r = a$, where $a > 0$.
Show that the area of the finite region lying within both curves is $a^2(4\pi - 3\sqrt{3})/12$.

4 Sketch the curve $r = a\cos 3\theta$, where $a > 0$, showing clearly the tangents to the curve at the pole.
Find the area of the finite region enclosed by one loop of the curve.

5 Write down an equation of the pair of straight lines tangential to the curve $(x^2 + y^2)^2 = 4(x^2 - y^2)$ at the origin and state the angle between them.
By conversion to polar coordinates, calculate the area of the finite region in the first quadrant bounded by the curve, the half line $\theta = \pi/4$ and the line $r = 2\sec\theta$.

6 Sketch the curve $r = 2(2 + \cos\theta)$ and find the area of the finite region bounded by the curve.

7 Sketch on the same diagram the curves

$$r = 2a(1 + \cos\theta)$$

and

$$r = a(3 + 2\cos\theta),$$

where a is a positive constant. Find the area of the region within which

$$2a(1 + \cos\theta) < r < a(3 + 2\cos\theta).$$

8 Sketch, on the same diagram, the circles given by $r = 2\cos\theta$, where $-\pi/2 \leqslant 0 \leqslant \pi/2$, and $r = 2\sin\theta$, where $0 \leqslant \theta \leqslant \pi$. Show that the circles intersect at the origin, and find the polar coordinates of the other point, P, at which the circles intersect.

Find also polar equations of the tangents to the two circles at P and a polar equation of the line joining P to the origin.

Calculate the area of the total region enclosed by the circles.

3.4 Use of matrices to represent linear transformations

The equations

$$\begin{aligned} x' &= ax + by + p, \\ y' &= cx + dy + q, \end{aligned} \tag{3.4}$$

representing a transformation which maps (x, y) onto (x', y'), can be expressed in matrix form thus:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix}. \tag{3.5}$$

Here $\begin{pmatrix} p \\ q \end{pmatrix}$ represents a translation, which for convenience we shall take to be $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ unless otherwise stated. For the equivalence of the matrix form to the original equations we require $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$, and we can say that the matrix $\mathbf{T} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represents the transformation $\mathbf{T}$.

Here, without ambiguity, we use the same symbol for the matrix and its associated transformation. Further, if $\mathbf{T}$ is equivalent to $\mathbf{T}_1$ followed by $\mathbf{T}_2$ (written $\mathbf{T}_2\mathbf{T}_1$—note the order), then we require

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \\ &= \begin{pmatrix} a_2a_1 + b_2c_1 & a_2b_1 + b_2d_1 \\ c_2a_1 + d_2c_1 & c_2b_1 + d_2d_1 \end{pmatrix}, \end{aligned} \tag{3.6}$$

as can be verified by direct substitution in equations. Cayley first used matrices

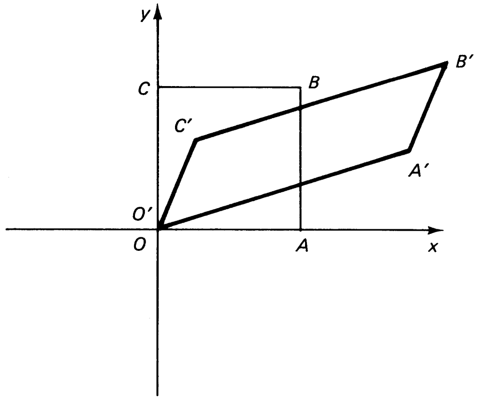

Fig. 3.9

in this context and evolved the rule for the 'multiplication' of matrices to give the correct result for composition of transformations.

3.5 Use of the unit square to identify transformations

If $A \equiv (1,0)$ and $C \equiv (0,1)$, $OABC$ is known as the *unit square* (Fig. 3.9). Since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix},$$

the vertices O, A and C of the unit square map onto O, (a, c) and (b, d) respectively under $\mathbf{T}$. Also, since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ c+d \end{pmatrix}$, the image $O'A'B'C'$ of of $OABC$ is clearly a parallelogram. Thus we can show in a figure the image of the unit square under any transformation represented by a 2×2 matrix, and

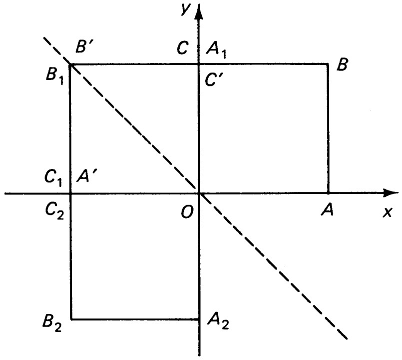

Fig. 3.10

hence in some cases identify the transformation. Conversely, given that such a transformation maps $(1,0)$ and $(0,1)$ onto (a,c) and (b,d) respectively, we can say that the matrix of the transformation is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Example 7 Find the matrix of the single transformation $\mathbf{T}$ equivalent to $\mathbf{T}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ followed by $\mathbf{T}_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, and interpret the result geometrically. Consider also the effect of $\mathbf{T}_2$ followed by $\mathbf{T}_1$.

Method (i): $\mathbf{T}_1$ maps $OABC$ onto $OA_1B_1C_1$ (Fig. 3.10) and is therefore an anti-clockwise rotation of 90° about O. $\mathbf{T}_2$ maps $OABC$ onto $OA_2B_2C_2$ and is therefore a reflection in $y + x = 0$. Reflecting $OA_1B_1C_1$ in $y + x = 0$, we obtain $OA'B'C'$, which is the reflection of $OABC$ in the y-axis. Similarly it can be shown that $\mathbf{T}_2$ followed by $\mathbf{T}_1$ is equivalent to reflection in the x-axis.

Method (ii): $\mathbf{T}_2\mathbf{T}_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{T}.$

$\mathbf{T}$ maps A onto $(-1,0)$ and C onto $(0,1)$, and is therefore reflection in the y-axis.

$$\mathbf{T}_1\mathbf{T}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \mathbf{S}.$$

$\mathbf{S}$ is reflection in the x-axis. This example illustrates the fact that composition of transformations is not commutative: $\mathbf{T}_1\mathbf{T}_2 \neq \mathbf{T}_2\mathbf{T}_1$ in general.

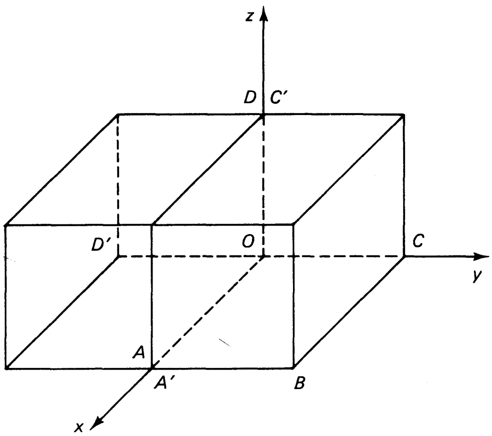

Fig. 3.11

Example 8 Interpret the transformation matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

From Fig. 3.11, showing the 'unit cube', the images of A, C and D are $(1, 0, 0)$, $(0, 0, 1)$ and $(0, -1, 0)$ respectively. Hence the matrix represents a clockwise rotation of 90° about Ox.

An *identity* or *unit matrix* represents an identity or 'stay-put' transformation under which every point maps onto itself. The identity matrices in two and three dimensions are $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ respectively. An identity matrix is usually written $\mathbf{I}$.

An *inverse* transformation undoes the effect of a previous transformation.

$\mathbf{T}^{-1}$ is the inverse of $\mathbf{T}$ $\quad\Leftrightarrow\quad \mathbf{T}^{-1}\mathbf{T} = \mathbf{T}\mathbf{T}^{-1} = \mathbf{I}$.

For example

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

showing that this reflection, and of course every reflection, is its own inverse.

The inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, as can easily be verified.

Exercise 3.5

1 The point P has position vector $\begin{pmatrix} x \\ y \end{pmatrix}$ referred to the origin O. When OP is rotated anti-clockwise through an angle θ about O, the position vector of P becomes $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$. Given that $\mathbf{T}_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, show that $\mathbf{T}_1 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Given that $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ is the reflection of $\begin{pmatrix} x \\ y \end{pmatrix}$ in the x-axis and $\mathbf{T}_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, find $\mathbf{T}_2$.

Given that $\mathbf{T}_\theta = \mathbf{T}_1\mathbf{T}_2\mathbf{T}_1{}^{-1}$ and $\mathbf{T}_\theta\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$, find $\mathbf{T}_\theta$. Deduce that $\begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$ is the reflection of $\begin{pmatrix} x \\ y \end{pmatrix}$ in the line $y = x\tan\theta$.

Given that $\mathbf{T}_3 = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$ and $\mathbf{T}_\phi = \mathbf{T}_3\mathbf{T}_2\mathbf{T}_3{}^{-1}$, show that $\mathbf{T}_\phi\mathbf{T}_\theta$ represents a rotation about O and find the angle of rotation.

2 (a) A linear transformation $\mathbf{T}_1$ of the plane has matrix $\begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}$. State the ratio of the area of the image under $\mathbf{T}_1$ of a finite region S to the area of S. Write down the matrix of the inverse transformation $\mathbf{T}_1^{-1}$ and find the coordinates of the point which is mapped onto the point $(1, -\frac{1}{2})$ by $\mathbf{T}_1$.

Find also equations of the two lines through the origin each of which is mapped onto itself by $\mathbf{T}_1$.

(b) A second transformation $\mathbf{T}_2$ has matrix $\begin{pmatrix} 4 & 2 \\ -2 & -1 \end{pmatrix}$. Explain why $\mathbf{T}_2$ has no inverse transformation and show that $\mathbf{T}_2$ maps all points of a certain line, whose equation should be given, onto $(1, -\frac{1}{2})$. Show also that $\mathbf{T}_2$ maps all points of the plane onto a line, and give the equation of this line.

3 Give a geometrical interpretation of the transformations given by the matrices $\mathbf{R}(\alpha)$ and $\mathbf{M}(k)$, where

$$\mathbf{R}(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, \qquad \mathbf{M}(k) = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}.$$

Show that the point Q with position vector $\begin{pmatrix} x \\ y \end{pmatrix}$ is transformed into the foot of the perpendicular from Q onto the line $y = x\tan\beta$ by means of the transformation with matrix $\mathbf{P}(\beta)$ where

$$\mathbf{P}(\beta) = \begin{pmatrix} \cos^2\beta & \cos\beta\sin\beta \\ \cos\beta\sin\beta & \sin^2\beta \end{pmatrix}.$$

Explain why, for a given value of β and a given point A with position vector $\begin{pmatrix} a \\ b \end{pmatrix}$, it is always possible to find values of α and k so that

$$\mathbf{M}(k)\mathbf{R}(\alpha)\begin{pmatrix} a \\ b \end{pmatrix} = \mathbf{P}(\beta)\begin{pmatrix} a \\ b \end{pmatrix}.$$

4 Find the 2×2 matrix $\mathbf{M}$ of the transformation which reflects each point P in the line $y = x$. Find also the 2×2 matrix $\mathbf{R}$ of the transformation which rotates OP anti-clockwise through a right angle about the origin O.

Show that under the transformation with matrix $\mathbf{RM}$ the image l_1 of the line $y = 3x + 4$ is the line $y + 3x = 4$.

Find the equation of the image l_2 of the line $y = 3x + 4$ under the transformation with matrix $\mathbf{MR}$, and show that the two images are parallel.

5 Matrices $\mathbf{A}$ and $\mathbf{B}$ represent linear transformations from $\mathbb{R}^2 \to \mathbb{R}^2$ and

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ -2 & 0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 3 & 5 \\ -6 & -10 \end{pmatrix}.$$

For each transformation, state the set of vectors which are transformed into the zero vector.

Show that the point $(t, 1 - 2t)$ lies on the line $2x + y = 1$ for all values of t.

The matrix $\mathbf{B}$ transforms the set of points

$$S_1 = \{(x, y) : 2x + y = 1\}$$

into the set of points S_2. Obtain, in a similar form, an expression for the set S_2.

By considering any member of the set S_2, determine the shortest distance between the line $2x + y = 1$ and a point whose coordinates are in the set S_2.

6 The transformation $\mathbf{T}_1$ operates on the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ according to the equation

$$\mathbf{T}_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Show that $\mathbf{T}_1$ corresponds to a reflection in the line $y = 1$.

A second transformation is given by

$$\mathbf{T}_2\begin{pmatrix} x \\ y \end{pmatrix} \equiv \frac{1}{25}\begin{pmatrix} -7 & 24 \\ 24 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

By considering

$$\mathbf{T}_2\begin{pmatrix} 3t + 4\lambda \\ 4t - 3\lambda \end{pmatrix},$$

or otherwise, show that $\mathbf{T}_2$ corresponds to a reflection in the line $3y = 4x$ and find the point Q which is invariant under both $\mathbf{T}_1$ and $\mathbf{T}_2$.

7 The transpose of a matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the matrix $\mathbf{M}^{\mathrm{T}} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, and $\mathbf{M}$ is said to be orthogonal when $\mathbf{M}^{\mathrm{T}}\mathbf{M} = \mathbf{I}$, where $\mathbf{I}$ is the unit matrix. Given that the matrix $\mathbf{N} = \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & \lambda \end{pmatrix}$ is orthogonal, find the value of λ. Describe geometrically the transformation of the x–y plane which is represented by $\mathbf{N}$.

Under a transformation $\mathbf{S}$ of the real plane into itself, a point $P \equiv (x, y)$ is mapped into the point

$$S(P) \equiv (ax + by,\ cx + dy).$$

Show that, when $\mathbf{M}$ is orthogonal, the distance between any two points P and Q is the same as that between their images $S(P)$ and $S(Q)$.

8 A point P lies in the plane of the cartesian axes Ox, Oy and has coordinates (x, y). The line OP is rotated, though an angle α in an anti-clockwise sense about an axis through the origin perpendicular to the plane, to a position OP' where P' has coordinates (x', y'). Show that

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Obtain the inverse of this rotation matrix, vertifying that it represents a clockwise rotation through an angle α.

The line OP' is similarly rotated in an anti-clockwise direction through an angle β to a position OP'', where P'' has coordinates (x'', y''). Find a single matrix which represents the rotation of OP to OP''. Hence verify that

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

9 State the matrix representations of the following transformations of the coordinate plane into itself:

(a) reflection in a line through O making an angle θ with Ox (θ being measured in the anti-clockwise direction),

(b) anti-clockwise rotation about O through an angle θ.

Linear transformations $\mathbf{T}_1$ and $\mathbf{T}_2$ are reflections of the plane in the lines $y = x$ and $y = x\tan(\pi/3)$ respectively. Write down the matrix representation of $\mathbf{T}_1$ and $\mathbf{T}_2$. Find the matrix representation of the combined transformation $\mathbf{T}_2\mathbf{T}_1$ and interpret the combined transformation geometrically.

A transformation $\mathbf{T}_3$ is a magnification from the origin by a factor 2; $\mathbf{T}_4$ is a translation with vector $\begin{pmatrix} -\sqrt{3} \\ -1 \end{pmatrix}$. Find the matrix of the transformation $\mathbf{T}_3\mathbf{T}_2\mathbf{T}_1$ and the image of the point (0, 1) under the transformation $\mathbf{T}_4\mathbf{T}_3\mathbf{T}_2\mathbf{T}_1$.

Miscellaneous exercise 3

1 Sketch the curve $r = a(1 + \sin\theta)$, where $a > 0$, and find the area of the region enclosed by the curve.

2 Sketch, for $-\pi/4 \leqslant \theta \leqslant \pi/4$, the curve with equation

$$r = a\cos 2\theta, \quad a > 0,$$

and find the area of the region for which $0 \leqslant r \leqslant a\cos 2\theta$, $-\pi/4 \leqslant \theta \leqslant \pi/4$.

By considering the stationary values of $\sin\theta\cos 2\theta$, or otherwise, find the polar coordinates of the points on the curve where the tangent is parallel to the initial line.

3 A straight line through the pole cuts the circle $r = a\cos\theta$ at P. The points A and B on this line are such that $AP = PB = a$. Sketch the locus of the points A and B, and show that its polar equation is

$$r = a(1 + \cos\theta).$$

Calculate the area of the finite region bounded by the arc of the locus for which $0 \leqslant \theta \leqslant \pi/4$, and by the half-lines $\theta = 0$, $\theta = \pi/4$.

4 Sketch the curves C_1 and C_2 whose polar equations are

$$C_1\colon r = 2\theta/\pi, \quad 0 \leqslant \theta \leqslant \pi/2,$$

$$C_2\colon r = \sin\theta, \quad 0 \leqslant \theta \leqslant \pi/2.$$

The half-line $\theta = \alpha$ meets the curve C_1 at the pole O and at P, and it meets C_2 at O and at Q. Find the limit as $\alpha \to 0$ of PQ/OP.

5 Given that $P_1(x_1, y_1)$ is the reflection of $P(x, y)$ in the line $y = x\tan\alpha$ and

$$\mathbf{M}_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix},$$

show that

$$\mathbf{M}_1 = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}.$$

Given also that $P_2(x_2, y_2)$ is the reflection of $P(x, y)$ in the line $y = x\tan\beta$,

$$\mathbf{M}_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad \text{and} \quad \mathbf{M}_2\mathbf{M}_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix},$$

show that $P_3(x_3, y_3)$ can be obtained by rotating OP about the origin O. State the angle through which OP would be rotated.

6 The linear transformation

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where $\mathbf{M}$ is a 3×3 matrix, maps the points with position vectors $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$ to the points with position vectors $\begin{pmatrix}3\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}$ respectively.

Write down the matrix $\mathbf{M}$ and find the inverse matrix $\mathbf{M}^{-1}$. Show that the transformation with matrix $\mathbf{M}$ maps points of the plane $x+y+z=0$ to points of the plane $x=y$. Verify that the inverse transformation with matrix $\mathbf{M}^{-1}$ maps points of the plane $x=y$ to points of the plane $x+y+z=0$.

7 Two matrices $\mathbf{S}_1$ and $\mathbf{S}_2$ are given by $\mathbf{S}_1 = \begin{pmatrix}1 & -1\\0 & 1\end{pmatrix}$ and $\mathbf{S}_2 = \begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}$.

(a) Show in separate diagrams the transformations effected by $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_2\mathbf{S}_1$ and $\mathbf{S}_1\mathbf{S}_2$, operating on the vector $\begin{pmatrix}x\\y\end{pmatrix}$.

(b) Show that $\mathbf{S}_1\mathbf{S}_2\mathbf{S}_1 = \mathbf{S}_2\mathbf{S}_1\mathbf{S}_2$.

(c) Apply the transformation $\mathbf{S}_1\mathbf{S}_2\mathbf{S}_1$ to the triangle with vertices $(1, 1)$, $(1, 3)$, $(2, 1)$ and show in the same diagram the triangle and its image. State the geometrical effect of the transformation.

8 $\mathbf{T} = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$ operates on the vector $\begin{pmatrix}x\\y\end{pmatrix}$. Name the transformation effected by each of (a) $\mathbf{T}\begin{pmatrix}x\\y\end{pmatrix}$, (b) $\mathbf{T}^2\begin{pmatrix}x\\y\end{pmatrix}$, (c) $\mathbf{T}^4\begin{pmatrix}x\\y\end{pmatrix}$.

9 State the transformation represented by each of the matrices:

(a) $\begin{pmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & -1\end{pmatrix}$, (b) $\begin{pmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{pmatrix}$, (c) $\begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix}$.

10 A is the point $(1, 0)$, B is the point $(0, 1)$ referred to the rectangular axes Ox, Oy. The transformation matrix $\begin{pmatrix}2 & 1\\0 & 2\end{pmatrix}$ is applied to the vertices of the triangle OAB. Show that the origin is invariant under this transformation.

Find the coordinates of the vertices of the image triangle $OA'B'$.

Find the transformation matrix which would convert the triangle $OA'B'$ back into the triangle OAB.

11 Show that the matrix $\begin{pmatrix}3 & 4\\-2 & -3\end{pmatrix}$ is equal to its own inverse. Explain geometrically.

12 Show that the matrix $\begin{pmatrix}2 & 4\\3 & 6\end{pmatrix}$ has no inverse. Explain geometrically.

4 Complex numbers

4.1 How numbers have developed

With the development of mathematics the concept of number has widened to meet the requirements for the solution of increasingly sophisticated problems. The process is illustrated by means of a Venn diagram (Fig. 4.1). We can use quadratic equations as models of the problems to be solved:

Integers (positive or negative) solve equations of the form

$$(x+1)(x-3)=0.$$

Rational numbers (ratios of integers) solve equations of the form

$$(3x+5)(2x-1)=0.$$

Irrational numbers, for example $\sqrt{7}$, solve equations of the form

$$x^2-2x-6=0.$$

So far we have used numbers which may all be called 'real'; but no real number satisfies an equation such as $x^2+6x+11=0$, equivalent to $(x+3)^2=-2$. To continue with this equation

$$x+3=\pm\sqrt{(-2)} \quad \Leftrightarrow \quad x=-3\pm\sqrt{(-2)}.$$

If we now write $\sqrt{(-1)}=\mathrm{i}$, we have obtained 'solutions' to the equation in the form $-3\pm \mathrm{i}\sqrt{2}$. Such numbers of the form $a+\mathrm{i}b$, where a and b are real, are called *complex numbers*; a is known as the *real* part and b as the *imaginary* part of $a+\mathrm{i}b$. This extension of the concept of number is useful, in fact for many purposes it is essential. However, we must ensure that complex numbers

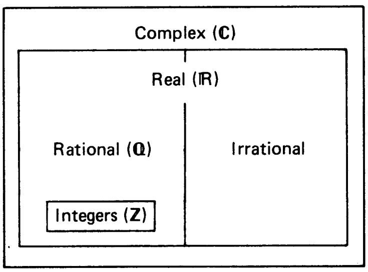

Fig. 4.1

obey the same structural laws as real numbers so that we can regard real numbers as special cases of complex numbers for which $b = 0$. (The set $\mathbb{R}$ of real numbers is a subset of the set $\mathbb{C}$ of complex numbers.) This is done by means of definitions which impose the laws of real algebra on complex numbers, together with the substitution $i^2 = -1$ when required.

4.2 Operations on complex numbers

Addition is defined on $\mathbb{C}$ so that

$$(a + ib) + (c + id) = (a + c) + i(b + d). \tag{4.1}$$

Thus addition on $\mathbb{C}$ is commutative since

$$(a + ib) + (c + id) = (c + id) + (a + ib),$$

associative since

$$[(a + ib) + (c + id)] + (f + ig) = (a + ib) + [(c + id) + (f + ig)],$$

and closed since

$$[(a + c) + i(b + d)] \in \mathbb{C}.$$

Subtraction is defined on $\mathbb{C}$ so that

$$(a + ib) - (c + id) = (a - c) + i(b - d); \tag{4.2}$$

subtraction is clearly not commutative or associative, but $\mathbb{C}$ is closed under subtraction.

Multiplication is defined on $\mathbb{C}$ so that

$$(a + ib)(c + id) = ac + i^2bd + i(ad + bc) = (ac - bd) + i(ad + bc); \tag{4.3}$$

multiplication on $\mathbb{C}$ is commutative, associative and closed.

Division is defined on $\mathbb{C}$ so that

$$\frac{a + ib}{c + id} = \frac{a + ib}{c + id} \times \frac{c - id}{c - id}$$

(compare to rationalising the denominator in real algebra; e.g.

$$\left.\frac{1}{3 - \sqrt{2}} = \frac{3 + \sqrt{2}}{7}\right)$$

$$\Rightarrow \quad \frac{a + ib}{c + id} = \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}$$

$$= \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2}. \tag{4.4}$$

Thus division on $\mathbb{C}$ is not commutative or associative, but closed. These structural properties are precisely the same as those of the four operations on $\mathbb{R}$.

Clearly it is not necessary to memorise the above results. We simply proceed as in real algebra, remembering to replace i^2 by -1 when it occurs.

Example 1 Express as a single complex number (i) $(7-3i)+(2+5i)$; (ii) $3(1-4i)-2(5-6i)$; (iii) $(4+5i)(2-i)$; (iv) $\dfrac{3-i}{1+2i}$.

(i) $(7+2)+i(5-3)=9+2i$.
(ii) $(3-10)-i(12-12)=-7$.
(iii) $8-5i^2+i(10-4)=8+5+6i=13+6i$.
(iv) $\dfrac{3-i}{1+2i}\times\dfrac{1-2i}{1-2i}=\dfrac{3+2i^2-(1+6)i}{1^2-4i^2}=\dfrac{1-7i}{5}=\dfrac{1}{5}-\dfrac{7}{5}i.$

Exercise 4.2

1 Express in the form $a+ib$, where $a, b\in\mathbb{R}$, each of the complex numbers

$$z_1=\frac{-2+i}{1-3i} \quad\text{and}\quad z_2=\frac{-3+i}{2+i}.$$

2 Express in the form $a+ib$, where $a, b\in\mathbb{Z}$,
(a) $(2-i)^3$; (b) $(13-i)/(3-i)$.

3 Given that $z=-1+3i$, express $z+\dfrac{2}{z}$ in the form $a+ib$ where $a, b\in\mathbb{R}$.

4 Given that $z_1=2+i$ and $z_2=1+2i$, express z_1/z_2 in the form $a+ib$, where $a, b\in\mathbb{R}$.

5 Given that

$$\frac{1}{x+iy}+\frac{1}{1+2i}=1,$$

where $x, y\in\mathbb{R}$, find x and y.

6 Given that $z_1=2$, $z_2=1+i\sqrt{3}$ and $z_3=1-i\sqrt{3}$,
(a) find $z_1z_2z_3$,
(b) show that $z_1z_2+z_1z_3+z_2z_3=8$.

7 Given that $z_1=2+i$, $z_2=1-2i$ and $\dfrac{1}{z_3}=\dfrac{1}{z_1}-\dfrac{1}{z_2}$, find z_3 in the form $a+ib$, where $a, b\in\mathbb{R}$.

4.3 Conjugate complex numbers

When we solve in complex algebra the quadratic equation ax^2+bx+c, *where a, b and c are real*, we obtain two solutions of the form $p+iq$ and $p-iq$. Note the requirement for $a, b, c\in\mathbb{R}$; for example

$$3x^2-2ix+1=0 \iff x=i \text{ or } -\tfrac{1}{3}i.$$

Two complex numbers of the form $p\pm iq$ form what is called a *conjugate pair*, and we say that each is the *conjugate* of the other. It is customary to use $z=x+iy$ to represent a complex variable and we may write $\mathrm{Re}(z)=x$, $\mathrm{Im}(z)=y$; then $x-iy$ is the conjugate of z, written z^*. Note that we express the quotient (4.4) without a complex denominator by multiplying the denominator $(c+id)$ by its conjugate $(c-id)$.

We define the complex number zero by saying

$$a + \mathrm{i}b = 0 \quad \Leftrightarrow \quad a = b = 0. \tag{4.5}$$

Thus

$$a + \mathrm{i}b = c + \mathrm{i}d \quad \Leftrightarrow \quad (a - c) + \mathrm{i}(b - d) = 0 \quad \Leftrightarrow \quad a = c, \quad b = d.$$

This means that a single equation in complex algebra is equivalent to two separate equations in real algebra.

Example 2 $3z = 7 + 4\mathrm{i} \quad \Leftrightarrow \quad 3(x + \mathrm{i}y) = 7 + 4\mathrm{i}$

$$\Leftrightarrow \quad 3x = 7, \quad 3y = 4 \quad \Leftrightarrow \quad x = 7/3, \quad y = 4/3.$$

The process of forming two separate equations involving only real numbers is called 'equating real and imaginary parts'.

Example 3 Solve for real values of x and y the equation

$$\frac{x}{1 + \mathrm{i}} - \frac{y}{2 - \mathrm{i}} = \frac{1 - 5\mathrm{i}}{3 - 2\mathrm{i}}$$

$$\Leftrightarrow \quad x(2 - \mathrm{i})(3 - 2\mathrm{i}) - y(1 + \mathrm{i})(3 - 2\mathrm{i}) = (1 - 5\mathrm{i})(1 + \mathrm{i})(2 - \mathrm{i})$$

$$\Leftrightarrow \quad x(4 - 7\mathrm{i}) - y(5 + \mathrm{i}) = (1 - 5\mathrm{i})(3 + \mathrm{i}) = (8 - 14\mathrm{i}).$$

Equating real and imaginary parts

$$\Leftrightarrow \quad 4x - 5y = 8$$

$$-7x - y = -14$$

$$\Leftrightarrow \quad x = 2, \quad y = 0.$$

The idea of a conjugate pair of complex numbers has an interesting and important application to the solution of algebraic equations. Suppose that $p + \mathrm{i}q$ $(q \neq 0)$ is a root of an algebraic equation $\mathrm{P}(x) = 0$, where $\mathrm{P}(x)$ is a polynomial of degree n, $ax^n + a_{n-1}x^{n-1} + \ldots + a_0$ *with real coefficients*. Then

$$a_n(p + \mathrm{i}q)^n + a_{n-1}(p + \mathrm{i}q)^{n+1} + \ldots + a_0 = 0.$$

The expansion of $(p + \mathrm{i}q)^r$, with $\mathrm{i}^2 = -1$, $\mathrm{i}^3 = -\mathrm{i}$, $\mathrm{i}^4 = 1$ etc., will be a complex number, and hence $\mathrm{P}(p + \mathrm{i}q)$ is a complex number, say $X + \mathrm{i}Y$.

$$\mathrm{P}(p + \mathrm{i}q) = 0 \quad \Leftrightarrow \quad X + \mathrm{i}Y = 0 \quad \Leftrightarrow \quad X = Y = 0 \qquad \text{by (4.5)}$$

$$\Leftrightarrow \quad X - \mathrm{i}Y = 0 \quad \Leftrightarrow \quad \mathrm{P}(p - \mathrm{i}q) = 0.$$

Hence $p + \mathrm{i}q$ is a root of $\mathrm{P}(x) = 0 \quad \Leftrightarrow \quad p - \mathrm{i}q$ is also a root. Thus the *complex* roots of an algebraic equation of degree n with real coefficients occur in conjugate pairs. Since in complex algebra such an equation has just n roots, this means that for instance a cubic equation must have either two complex roots

or none, and hence must have one or three real roots; a quartic has four complex roots or two or none, and thus must have either no real roots or two or four, and so on.

Note that the above discussion can be extended to show that: 'In *any* identity relating complex numbers, i can be changed into $-$i and the identity remains true'.

Example 4 Express in the form $a + b\mathrm{i}$,

$$\sqrt{[(1 + \mathrm{i})(17 + 7\mathrm{i})]}.$$

$$a + b\mathrm{i} = \sqrt{[(1 + \mathrm{i})(17 + 7\mathrm{i})]}$$

$$\Rightarrow \quad (a + b\mathrm{i})^2 = (1 + \mathrm{i})(17 + 7\mathrm{i}) = 10 + 24\mathrm{i}. \qquad \text{(i)}$$

Replacing i by $-$i,

$$(a - b\mathrm{i})^2 = 10 - 24\mathrm{i}. \qquad \text{(ii)}$$

Multiplying (i) and (ii),

$$(a^2 + b^2)^2 = (10 + 24\mathrm{i})(10 - 24\mathrm{i}) = 676$$

$$\Leftrightarrow \quad a^2 + b^2 = 26 \qquad [a^2 + b^2 > 0].$$

Equating real and imaginary parts in (i)

$$a^2 - b^2 = 10$$

$$\Leftrightarrow \quad a^2 = 18, \quad b^2 = 8 \quad \Leftrightarrow \quad a = 3\sqrt{2}, \quad b = 2\sqrt{2}.$$

[A root of a complex number may be found in another way—see p. 67.]

Example 5 Verify that $1 - \mathrm{i}$ is a root of the equation $z^4 + 3z^2 - 6z + 10 = 0$, and hence solve the equation completely.

Note that here, conventionally, we express the equation in terms of z, since z may be complex.

$$(1 - \mathrm{i})^4 + 3(1 - \mathrm{i})^2 - 6(1 - \mathrm{i}) + 10 = (-2\mathrm{i})^2 - 6\mathrm{i} - 6 + 6\mathrm{i} + 10$$
$$= -4 - 6 + 10 = 0.$$

Hence $1 - \mathrm{i}$ is a root, and hence also is $1 + \mathrm{i}$,

$$(z - 1 + \mathrm{i})(z - 1 - \mathrm{i}) \equiv z^2 - 2z + 2.$$

$$z^4 + 3z^2 - 6z + 10 \equiv (z^2 - 2z + 2)(z^2 + 2z + 5)$$

$$\Rightarrow \quad \text{remaining roots are } -1 \pm 2\mathrm{i}.$$

4.4 The Argand diagram

The French mathematician Argand realised that the coordinate system introduced by Descartes, in which two perpendicular axes are used as number lines

to describe the position of a point in their plane, provides a convenient way of representing complex numbers diagrammatically. The x- and y-axes of the cartesian system become 'real' and 'imaginary' axes along which the real and imaginary parts respectively of complex numbers are measured; so that the complex number $a + ib$ is represented in an Argand diagram by the point with cartesian coordinates (a, b), and the complex number z will be represented by the point (x, y). Many results in complex algebra can be conveniently illustrated or obtained by means of an Argand diagram.

Example 6 Illustrate addition and subtraction of complex numbers in an Argand diagram.

(i) $(a + ib) + (c + id) = (a + c) + i(b + d)$ (see Fig. 4.2).
Anyone familiar with vectors will see that the figure is equivalent to a vector diagram illustrating the 'parallelogram rule' for the addition of vectors,

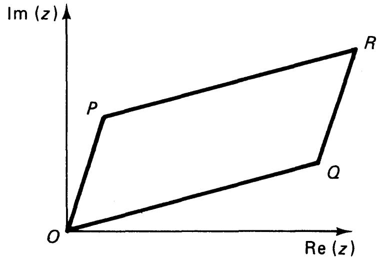

Fig. 4.2

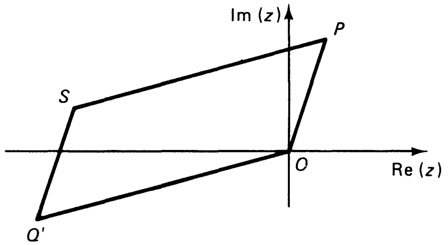

Fig. 4.3

$$\overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OR}.$$

Clearly we can regard the complex number $a + ib$ as being represented in the Argand diagram *either* by the point $P(a, b)$ *or* by the displacement $\overrightarrow{OP}$.
(ii) $(a + ib) - (c + id) = (a - c) + i(b - d)$ (see Fig. 4.3).
Notice that if Q represents $c + id$, then $-(c + id)$ is represented by Q', the reflection of Q in O. We say that $\overrightarrow{OP} - \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{OQ'} = \overrightarrow{OS}$.

4.5 Modulus and argument

Having seen how cartesian coordinates can be adapted to represent the real and imaginary parts of complex numbers, we might consider whether the alternative system of polar coordinates has any relevance to complex algebra. In this case we adapt the complex numbers to the coordinates rather than vice versa.

If $(x, y) \equiv (r, \theta)$, from Fig. 4.4,

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$\Leftrightarrow \quad z = x + iy = r(\cos\theta + i\sin\theta),$$

$$r = \sqrt{(x^2 + y^2)}, \quad \tan\theta = y/x.$$

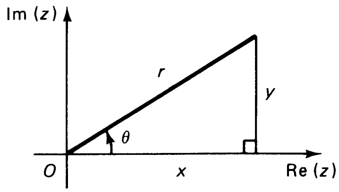

Fig. 4.4

Thus polar coordinates suggest an alternative way of writing complex numbers, which turns out to be extremely useful. Although in using polar coordinates we sometimes allow r to be positive or negative, in complex algebra r is defined (for convenience) as $+\sqrt{(x^2+y^2)}$ and is called the *modulus* of x, written

$$|z| = r = +\sqrt{(x^2+y^2)}.$$

θ is called the *argument* of z, written $\arg z$, and is deliberately *not* defined to be single valued. In Figs. 4.5, (a)–(d),

$$\theta = \arg z = \alpha + 2n\pi \quad (n = 0, \pm 1, \pm 2, \ldots),$$

i.e. for $n \in \mathbb{Z}$. For convenience we define the *principal value* of θ as the value θ_P for which $-\pi < \theta_P \leqslant \pi$; for example if $\theta = 5\pi/2$, $\theta_P = \pi/2$.

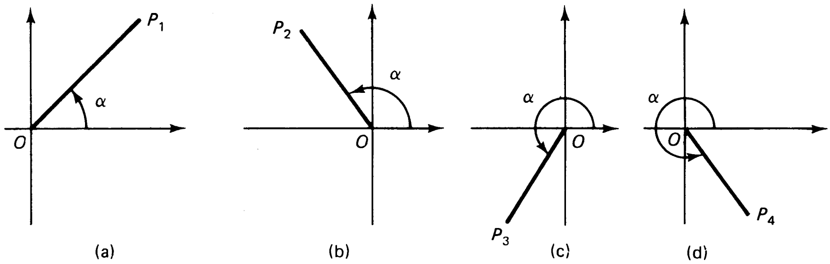

Fig. 4.5

Note that $zz^* = x^2 + y^2 = |z|^2$. It follows that if $f(z) = X + iY$ is any function of z,

$$|f(z)| = \sqrt{(X^2+Y^2)}.$$

Here $[f(z)]^* = f^*(z^*)$; that is we must not only change z into z^* but change i occurring in any numerical coefficient into $-$i.

Example 7

(i) $|z-3i|^2 = (z-3i)(z^*+3i)$

$$= zz^* + 3i(z - z^*) - 9i^2$$

$$= x^2 + y^2 - 6y + 9.$$

(ii) If $f(z) = \dfrac{z - 6i}{z^2 + 2iz}$, $[f(z)]^* = \dfrac{z^* + 6i}{z^{*2} - 2iz^*}$.

It is sometimes convenient to write $(\cos\theta + i\sin\theta)$ as $\operatorname{cis}\theta$. The notation $[r, \theta]$ is often used for the complex number with modulus r and argument θ.

Example 8 Express in modulus–argument form (a) $3 + 4i$, (b) $-3 + 4i$, (c) $-3 - 4i$, (d) $3 - 4i$.

The modulus of each of these complex numbers is $+\sqrt{(3^2 + 4^2)} = 5$. For (a) and (c) the argument is $\arctan(4/3)$; for (b) and (d) the argument is $\arctan(-4/3)$, but these facts are not sufficient to determine the arguments. We advise the use of a figure to determine the positions of the points representing complex numbers on an Argand diagram by means of their 'coordinates'. In this example Figs. 4.5 (a)–(d) illustrate the four cases and we obtain (a) $[5, 53{\cdot}1°]$, (b) $[5, 126{\cdot}9°]$, (c) $[5, 233{\cdot}1°]$, (d) $[5, 306{\cdot}9°]$. Note that the principal values of the arguments in (c) and (d) are $-126{\cdot}9°$ and $-53{\cdot}1°$ respectively.

Example 9 Express in the form $a + ib$ the complex numbers
(a) $[3, 0]$, (b) $[2, \frac{1}{4}\pi]$, (c) $[10, \frac{1}{3}\pi]$, (d) $[1, -\frac{1}{2}\pi]$.

(a) $3(\cos 0 + i\sin 0) = 3$; (b) $2(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi) = \sqrt{2}(1 + i)$;
(c) $10(\cos\frac{1}{3}\pi + i\sin\frac{1}{3}\pi) = 5(1 + i\sqrt{3})$; (d) $\cos(-\frac{1}{2}\pi) + i\sin(-\frac{1}{2}\pi) = -i$.

Example 10 (a) Given that $z_1 = \dfrac{2 - i}{2 + i}$, $z_2 = \dfrac{2i - 1}{1 - i}$, express z_1 and z_2 in the form $a + ib$.
(b) Sketch an Argand diagram showing points P and Q representing the complex numbers $5z_1 + 2z_2$ and $5z_1 - 2z_2$ respectively.

$$\text{(a)}\quad z_1 = \frac{2 - i}{2 + i} \times \frac{2 - i}{2 - i} = \frac{4 - 4i + i^2}{4 - i^2} = \frac{3 - 4i}{5} = \frac{3}{5} - \frac{4}{5}i.$$

$$z_2 = \frac{2i - 1}{1 - i} \times \frac{1 + i}{1 + i} = \frac{-1 + i + 2i^2}{1 - i^2} = \frac{-3 + i}{2} = -\frac{3}{2} + \frac{1}{2}i.$$

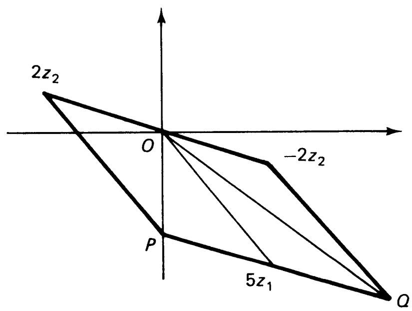

Fig. 4.6

(b) $5z_1 + 2z_2 = 3 - 4i + (-3 + i) = -3i.$

$5z_1 - 2z_2 = 3 - 4i - (-3 + i) = 6 - 5i.$

In Fig. 4.6, notice that multiplication of a complex number by a real number, for example, $5z_1$, $2z_2$, changes the modulus but not the argument of the complex number. Compare multiplication of a vector by a scalar, for example, $5\mathbf{z}_1$, $2\mathbf{z}_2$.

Example 11 Describe the set of points $[r, \theta]$ for which (a) $r = 2$, (b) $\theta = \pi$.

(a) This is the set of points at distance 2 from the origin; that is the circle with centre O and radius 2.
(b) These points lie on the negative part of the real axis.

Example 12 Illustrate multiplication on complex numbers in an Argand diagram.

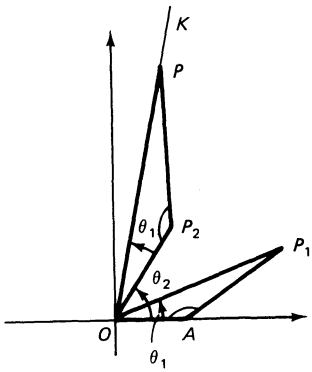

Fig. 4.7

If P_1, P_2 represent $[r_1, \theta_1]$, $[r_2, \theta_2]$ on an Argand diagram, Fig. 4.7, we require a construction for P representing $[r_1 r_2, \theta_1 + \theta_2]$. Clearly P lies on OK making the angle $\theta_1 + \theta_2$ with the real axis. Since $OP : OP_2 = r_1 r_2 : r_2 = r_1 : 1$ and $P\hat{O}P_2 = P_1\hat{O}A = \theta_1$, the triangle POP_2 is similar to triangle P_1OA, where A represents $[1, 0]$. Hence to obtain P we construct $O\hat{P}_2P = O\hat{A}P$, so that P_2P meets OK at P.

4.6 Distance and direction on the Argand diagram

If P_1, P_2 represent the complex numbers z_1, z_2 respectively on the Argand diagram, Fig. 4.8, then

$$z_2 - z_1 = (x_2 - x_1) + i(y_2 - y_1)$$

$$\Rightarrow \quad P_1N = x_2 - x_1 = \mathrm{Re}(z_2 - z_1),$$

$$P_2N = y_2 - y_1 = \operatorname{Im}(z_2 - z_1)$$

$$\Rightarrow \quad P_1P_2 = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]} = |z_2 - z_1|,$$

$$P_2\hat{P}_1N = \arg(z_2 - z_1).$$

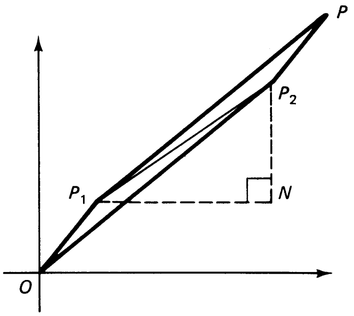

Fig. 4.8

Again there is a close resemblance to a vector diagram in which if $\overrightarrow{OP_1} = \mathbf{z}_1$, $\overrightarrow{OP_2} = \mathbf{z}_2$, then $P_1P_2 = |\mathbf{z}_2 - \mathbf{z}_1|$. These results will frequently be found useful.

Referring again to Fig. 4.8, P represents the complex number $z_1 + z_2$. Since $OP \leqslant OP_1 + P_1P$,

$$|z_1 + z_2| \leqslant |z_1| + |z_2|.$$

Equality occurs only when OP_1P_2P is straight line; that is when

$$\arg z_1 = \arg z_2 = \arg(z_1 + z_2).$$

Clearly the result can be extended by induction for the n complex numbers $z_1, z_2, \ldots, z_n$, for which

$$|z_1 + z_2 + \cdots + z_n| \leqslant |z_1| + |z_2| + \cdots + |z_n|$$

with equality only when $\arg z_1 = \arg z_2 = \cdots = \arg z_n$.

Example 13 If z_1, z_2 are complex numbers, prove that
(a) $|z_1 + z_2| \geqslant |z_1| - |z_2|$; (b) $|z_1 - z_2| \geqslant |z_1| - |z_2|$.

In Fig. 4.8,

(a) $OP + PP_1 \geqslant OP_1 \quad \Rightarrow \quad |z_1 + z_2| + |z_2| \geqslant |z_1|$

$\qquad\qquad\qquad\qquad\quad \Leftrightarrow \quad |z_1 + z_2| \geqslant |z_1| - |z_2|$;

(b) $P_1P_2 + PP_2 \geqslant PP_1 \quad \Rightarrow \quad |z_1 - z_2| + |z_2| \geqslant |z_1|$

$\qquad\qquad\qquad\qquad\quad \Leftrightarrow \quad |z_1 - z_2| \geqslant |z_1| - |z_2|$.

Exercise 4.6

1 Given that $2 + i$ is a root of the cubic equation

$$z^3 - 11z + 20 = 0,$$

find the other two roots.

2 Find $|z|$ and $\arg z$ for each of the complex numbers z given by
(a) $12 - 5i$, (b) $(1 + 2i)/(2 - i)$,
giving the argument in degrees, to the nearest degree, such that $-180° < \arg z \leqslant 180°$.

3 Find the roots z_1 and z_2 of the quadratic equation

$$z^2 - 3(1 + i)z + 5i = 0,$$

expressing your answers in the form $c + id$, where $c, d \in \mathbb{R}$. Explain why z_1 and z_2 are not conjugate complex numbers.

Plot z_1 and z_2 on a sketch of the Argand diagram and, preferably without first solving the equation, plot on the same diagram the roots z_3, z_4 of the quadratic equation

$$z^2 - 3(1 - i)z - 5i = 0.$$

4 Given that $z_1 = \frac{1}{2}(-1 + i\sqrt{3})$, find $|z_1|$ and $\arg z_1$.

Represent z_1, $1/z_1$ and $(z_1 - 1/z_1)$ by vectors on an Argand diagram.

Find, in algebraic form, the three roots of the equation $z^3 - 1 = 0$. Represent these roots by points on an Argand diagram, indicating their polar coordinates.

5 Obtain, in the form $a + ib$, where $a, b \in \mathbb{R}$, the four complex roots of the equation

$$z^4 + 6z^2 + 25 = 0$$

and plot their positions on an Argand diagram. Show that these four points lie on a circle C and find the radius of C.

6 Find the modulus of each of the complex numbers z_1 and z_2, where

$$z_1 = 1 + 7i \quad \text{and} \quad z_2 = -4 - 3i.$$

Hence show that the modulus of z_1/z_2 is $\sqrt{2}$. Find also the argument of z_1/z_2.

7 Determine all pairs of values of p, q, where $p, q \in \mathbb{R}$, for which $1 + i$ is a root of the equation

$$z^3 + pz^2 + qz - pq = 0.$$

8 Solve the equation

$$z^4 + 8z^2 + 16z + 20 = 0,$$

given that $1 + 3i$ is a complex root.

9 Three of the five roots of an equation of the fifth degree in z with real coefficients are $z = 1$ and $z = i \pm 1$. Show that the equation can be written

$$z^5 - z^4 + 4z - 4 = 0$$

10 Given that $|z - a| \leqslant \frac{1}{2}|a|$, show, geometrically or otherwise, that $|z| \geqslant \frac{1}{2}|a|$.

4.7 Loci on the Argand diagram

When working with a complex variable z it is often necessary to specify some restriction on the possible values of z, and to identify the corresponding set of points, or locus, on an Argand diagram.

Example 14 Identify each of the loci:

(a) $|z| = 6$, (b) $|2z - 3| = 4$, (c) $|z + 1| = |z - 3|$,

(d) $\arg z = \pi/6$, (e) $|z - 1| = 3|z + 2|$, (f) $|z + 1| + |z - 1| = 8$,

(g) $zz^* = 4$.

The best way of identifying a locus is to interpret its equation in such a way that the geometry of the figure enables us to recognise it. If we cannot do this, or if it is essential to find the cartesian equation of the locus, we put $z = x + iy$ and obtain a real relation between x and y, which is the equation of the locus referred to the real and imaginary axes as x- and y-axes respectively.

(a) $|z|$ is the distance from the origin of the point representing z. Since this is constant, the locus is the circle with centre O and radius 6.

(b) $|2z - 3| = 4 \Leftrightarrow |z - 3/2| = 2$. But $|z - 3/2|$ is the distance of z from $(3/2, 0)$; hence the locus of z is the circle with centre $(3/2, 0)$ and radius 2.

(c) The equation states that the distance of z from $(-1, 0)$ equals its distance from $(3, 0)$. Hence the locus of z is the perpendicular bisector (mediator) of the line segment joining $(-1, 0)$ and $(3, 0)$; that is the line $x = 1$ or $\mathrm{Re}(z) = 1$.

Alternatively,

$$|z + 1| = |z - 3| \Leftrightarrow |x + iy + 1| = |x + iy - 3|$$

$$\Leftrightarrow (x + 1)^2 + y^2 = (x - 3)^2 + y^2 \Leftrightarrow 2x + 1 = -6x + 9 \Leftrightarrow x = 1,$$

or

$$|z + 1|^2 = |z - 3|^2 \Leftrightarrow (z + 1)(z^* + 1) = (z - 3)(z^* - 3)$$

$$\Leftrightarrow z + z^* = 2 \Leftrightarrow x = 1$$

as before.

(d) If P is a point on the locus, then OP makes an angle $\pi/6$ with the positive real axis. Hence the locus of P is the *half-line* from O at an angle $\pi/6$ with $\mathrm{Re}(z) = 0$; that is, with the x-axis.

(e) The equation states that the ratio of the distances of z from $(1, 0)$ and $(-2, 0)$ is 3. A knowledge of pure geometry makes it possible to say that the locus is an (Apollonius) circle, but it does not directly give the equation of the circle.

$$|z - 1| = 3|z + 2| \Leftrightarrow |x + iy - 1| = 3|x + iy + 2|$$

$$\Leftrightarrow (x - 1)^2 + y^2 = 9[(x + 2)^2 + y^2]$$

$$\Leftrightarrow 8(x^2 + y^2) + 38x + 35 = 0.$$

This is clearly the equation of a circle.

Alternatively we can find the same equation by using the relation

$$|z - 1|^2 = 9|z + 2|^2 \Leftrightarrow (z - 1)(z^* - 1) = 9(z + 2)(z^* + 2).$$

(f) From the result on p. 21, it follows that the locus is an ellipse with foci $(1, 0)$ and $(-1, 0)$, since the equation states that the sum of the distances of z

from these two points is constant; the eccentricity of the ellipse is $\dfrac{1-(-1)}{8} = \dfrac{1}{4}$.

To obtain the equation of the ellipse in its simplest form involves some awkward algebra. The geometrical method is much simpler.

(g) $zz^* = 4 \quad \Leftrightarrow \quad (x + \mathrm{i}y)(x - \mathrm{i}y) = 4 \quad \Leftrightarrow \quad x^2 + y^2 = 4.$

This is the equation of the circle with centre O and radius 2.

Example 15 Identify the loci on the Argand diagram given by

(i) $|z - 1| + |z + 1| = 4$,

(ii) $|z - 1| + |z + 1| = 2$,

(iii) $\arg(z - 1) - \arg(z + 1) = \frac{1}{2}\pi$,

(iv) $\arg(z - 1) + \arg(z + 1) = \pi$,

(v) $\dfrac{|z-1|}{|z+1|} = 2.$

(i) The sum of the distances of z from $(1, 0)$ and $(-1, 0)$ is constant and greater than 2; hence the locus is the ellipse with foci $(1, 0)$ and $(-1, 0)$ and eccentricity $\frac{2}{4} = \frac{1}{2}$.

(ii) The sum of the distances of z from $(1, 0)$ and $(-1, 0)$ is equal to the distance between these points. Hence the locus of z is the portion of the real axis between $(1, 0)$ and $(-1, 0)$ inclusive.

(iii) The points $A(1, 0)$ and $B(-1, 0)$ must subtend a right angle at the point representing z; but since $\arg(z - 1) > \arg(z + 1)$, z must lie on the semicircle with AB as diameter and for which $y > 0$.

(iv) This is the same locus as (ii).

(v) $$\begin{aligned} |x + \mathrm{i}y - 1| = 2|x + \mathrm{i}y + 1| \quad &\Leftrightarrow \quad (x - 1)^2 + y^2 = 4[(x + 1)^2 + y^2] \\ &\Leftrightarrow \quad 3(x^2 + y^2) + 10x + 3 = 0. \end{aligned}$$

This is a circle.

Exercise 4.7

1 (a) Given that $z = 1 + \mathrm{i}\sqrt{3}$, find $|z|$ and $|z^5|$. Find also the values of $\arg z$ and $\arg(z^5)$ lying between $-\pi$ and π. Show that $\mathrm{Re}(z^5) = 16$ and find the value of $\mathrm{Im}(z^5)$.

(b) Draw on the Argand diagram the line $|z| = |z - 4|$ and the half-line $\arg(z - \mathrm{i}) = \pi/4$. Hence, or otherwise, find the complex number that satisfies both equations.

2 Express in the form $|z - a - \mathrm{i}b| = R$, where $a, b \in \mathbb{R}$, the equation of the circle on the Argand diagram which passes through the points given by $z = 4$, $z = 2\mathrm{i}$ and $z = 4 + 2\mathrm{i}$.

3 By shading in three separate Argand diagrams, show the regions in which the point representing z can lie when

(a) $|z| > 3$,

(b) $|z - 2| < |z - 4|$,

(c) $0 < \arg(z + 3) < \pi/6$.

Shade in another Argand diagram the region in which z can lie when all these inequalities apply.

4 Sketch on the Argand diagram the sets A and B where

$$A = \{z: |z| = 6, z \in \mathbb{C}\},$$
$$B = \{z: |z+9| = 9, z \in \mathbb{C}\}.$$

Shade the finite region for which $|z+9| < 9$ and $|z| > 6$.
Show that $n\{A \cap B\} = 2$.
Given that $A \cap B \equiv \{z_1, z_2\}$ where $\operatorname{Im}(z_1) > \operatorname{Im}(z_2)$, find
(a) $z_1 z_2$, (b) $z_1 + z_2$, (c) z_1/z_2.

5 The points A and B on the Argand diagram represent the complex numbers z_1 and z_2 respectively, where $0 < \arg z_2 < \arg z_1 < \pi/2$. State geometrical constructions to find the points C and D representing $z_1 + z_2$ and $z_1 - z_2$ respectively.
If $\arg(z_1 - z_2) - \arg(z_1 + z_2) = \pi/2$, prove that $|z_1| = |z_2|$.

6 Sketch on an Argand diagram the circle $|z-3| = 2$. State the greatest value of $|z|$ when $|z-3| = 2$.

7 The complex number $z = x + iy$ is such that

$$\frac{z+3}{z-4i} = ki$$

where $k \in \mathbb{R}$. Show that

$$x(x+3) + y(y-4) = 0.$$

When z is represented on an Argand diagram by the point P, show that when k varies P moves on a circle. Find the radius of this circle and the complex number corresponding to its centre.

8 The complex numbers z_1 and z_2 satisfy the equation

$$z_2{}^2 - z_1 z_2 + z_1{}^2 = 0.$$

Find z_2/z_1, given that its imaginary part is positive.
If $z_1 = a + ib$, where $a, b \in \mathbb{R}$, show that

$$z_2 = \tfrac{1}{2}(a - b\sqrt{3}) + \tfrac{1}{2}(b + a\sqrt{3})i.$$

On an Argand diagram, the points A and B represent z_1 and z_2, respectively. Show that the triangle OAB is equilateral.

9 (a) Given that $|z_1| = |z_2|$, where z_1 and z_2 are distinct complex numbers, prove that the complex number $(z_1 + z_2)/(z_1 - z_2)$ is purely imaginary.
(b) Find the modulus and argument of each root of the equation $z^3 + 27 = 0$ and show the three roots on an Argand diagram.
(c) Show that the points on the Argand diagram for which

$$|z-4| = 2|z-1|$$

all lie on the circle $|z| = 2$.

10 If z is the complex number $x + iy$, determine the loci in the z-plane defined by the following equations:
(a) $|z-1| = 3$, (b) $|z+i| = |z-2|$,
(c) $|z-1| = 2|z+1|$, (d) $|z-1| = \operatorname{Re}(z)$.

11 (a) Show that there are two complex numbers z_1 and z_2 which satisfy the equation

$$3zz^* + 2(z - z^*) = 29 + 12i,$$

where z^* denotes the conjugate of z. If points A and B represent z_1 and z_2 respectively on the Argand diagram (z-plane) and C represents i, show that $A\hat{C}B = \pi/2$.
(b) Shade on the z-plane the domain for which the following inequalities are all satisfied:

$$1 < |z| < 2, \quad \mathrm{Im}(z) > 1, \quad \pi/4 < \arg z < \pi/2.$$

Given that $\arg(z + \mathrm{i}) = \pi/4$ and $|z + 1| = 2$, show that z is real.

12 Sketch on an Argand diagram the set of points satisfying both

$$|z| < |z - 1| \quad \text{and} \quad -\tfrac{1}{4}\pi < \arg z < \tfrac{1}{4}\pi.$$

13 Interpret geometrically the following loci in the z-plane:
(a) $|z - 2| = |z + 1|$;
(b) $|z - 2| + |z + 1| = 4$;
(c) $\arg\left(\dfrac{z-2}{z+1}\right) = \dfrac{\pi}{4}$.

4.8 Product and quotient for complex numbers in modulus–argument form

The product $[r_1, \theta_1] \times [r_2, \theta_2]$ is, by definition, obtained from 'ordinary multiplication' with $\mathrm{i}^2 = -1$. Thus

$$\begin{aligned} & r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1) \times r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2) \\ &= r_1 r_2\Big((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + (\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\mathrm{i}\Big) \\ &= r_1 r_2\Big(\cos(\theta_1 + \theta_2) + \mathrm{i}\sin(\theta_1 + \theta_2)\Big). \end{aligned}$$

Hence we can formulate the simple rule: 'The product of two complex numbers in modulus–argument form is obtained by multiplying the moduli and adding the arguments'.

Example 16 (i) $[3, 35°] \times [5, 61°] = [15, 96°]$;
(ii) $[6, \pi/3] \times [2, -\pi/3] = [12, 0]$.

Using our previous method for division, $[r_1, \theta_1] \div [r_2, \theta_2]$

$$\begin{aligned} &= \frac{r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)}{r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)} \times \frac{(\cos\theta_2 - \mathrm{i}\sin\theta_2)}{(\cos\theta_2 - \mathrm{i}\sin\theta_2)} \\ &= \frac{r_1[(\cos(\theta_1 - \theta_2) + \mathrm{i}\sin(\theta_1 - \theta_2)]}{r_2} \\ &= [(r_1/r_2), (\theta_1 - \theta_2)]. \end{aligned}$$

Hence the rule for division: 'Divide the moduli and subtract the arguments'. Note that the argument rules for multiplication and division may not necessarily give principal values.

Example 17 (i) $[18, 53°] \div [3, 81°] = [6, -28°]$;
(ii) $[10, \pi/2] \div [2, \pi/3] = [5, \pi/6]$.

Example 18 Given that z_1 and z_2 are non-zero complex numbers such that $|z_1 + z_2| = |z_1 - z_2|$, explain, by drawing an Argand diagram, why

$$\arg\left(\frac{z_1}{z_2}\right) = \pm\frac{\pi}{2}.$$

In Fig. 4.8,

$$OP = |z_1 + z_2|, \quad P_1P_2 = |z_1 - z_2|.$$

$$|z_1 + z_2| = |z_1 - z_2| \quad \Leftrightarrow \quad OP_1PP_2 \text{ is a rectangle}$$

(parallelogram with diagonals equal)

$$\Leftrightarrow \quad P_1\hat{O}P_2 = 90° \quad \Leftrightarrow \quad \arg(z_1/z_2) = \pm\pi/2.$$

The $\pm$ arises since P_1 and P_2 may be interchanged.

4.9 De Moivre's Theorem

The rule for multiplication established in §4.8 can clearly be applied to a succession of products:

$$[r_1, \theta_1] \times [r_2, \theta_2] \times \ldots \times [r_n, \theta_n] = [r_1 r_2 \ldots r_n, (\theta_1 + \theta_2 + \ldots + \theta_n)].$$

When

$$r_1 = r_2 = \ldots = r_n = 1 \quad \text{and} \quad \theta_1 = \theta_2 = \ldots = \theta_n = \theta,$$

we have

$$[1, \theta]^n = [1, n\theta] \quad \text{or} \quad (\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta, \quad n \in \mathbb{Z}^+.$$

This is the simplest case of a more general result known as *de Moivre's theorem.*

From the above,

$$(\cos\theta + \mathrm{i}\sin\theta)^{-n} = \frac{1}{(\cos\theta + \mathrm{i}\sin\theta)^n} = \frac{1}{\cos n\theta + \mathrm{i}\sin n\theta}$$

$$= \frac{\cos n\theta - \mathrm{i}\sin n\theta}{\cos^2 n\theta + \sin^2 n\theta} = \cos(-n\theta) + \mathrm{i}\sin(-n\theta).$$

Hence de Moivre's theorem holds for all $n \in \mathbb{Z}$.

Example 19 Simplify $\dfrac{[\cos(7\pi/11) + \mathrm{i}\sin(7\pi/11)]^5}{[\cos(3\pi/11) - \mathrm{i}\sin(3\pi/11)]^3}$.

The expression $= \dfrac{[\operatorname{cis}(7\pi/11)]^5}{[\operatorname{cis}(-3\pi/11)]^3} = \dfrac{\operatorname{cis}(35\pi/11)}{\operatorname{cis}(-9\pi/11)}$

$$= \operatorname{cis}(44\pi/11) = \cos 4\pi + \mathrm{i}\sin 4\pi = 1.$$

Now consider $(\cos\theta + \mathrm{i}\sin\theta)^{p/q}$, where $p, q \in \mathbb{Z}$.

$$(\cos\theta + \mathrm{i}\sin\theta)^{p/q} = r(\cos\phi + \mathrm{i}\sin\phi) \quad \Rightarrow \quad (\operatorname{cis}\theta)^p = r^q(\operatorname{cis}\phi)^q$$

$$\Rightarrow \quad r^q = 1 \quad \Rightarrow \quad r = 1 \quad (r > 0)$$

$$\Rightarrow \quad \cos p\theta + \mathrm{i}\sin p\theta = \cos q\phi + \mathrm{i}\sin q\phi$$

by de Moivre's theorem. Equating real and imaginary parts

$$\cos p\theta = \cos q\phi, \quad \sin p\theta = \sin q\phi$$

$$\Rightarrow \quad q\phi = p\theta + 2k\pi, \quad \text{where } k \in \mathbb{Z},$$

$$\Rightarrow \quad \phi = p\theta/q + 2k\pi/q.$$

The values 0, 1, 2, ..., $(q-1)$ give q different values for $\operatorname{cis}\phi$; but $k = q$ gives $\phi = p\theta/q + 2\pi$ and so the same value of $\operatorname{cis}\phi$ as for $k = 0$. Hence $(\cos\theta + \mathrm{i}\sin\theta)^{p/q}$ has just q values, given by

$$\operatorname{cis}(p\theta/q + 2k\pi/q), \quad k = 0, 1, 2, \ldots, (q-1).$$

Example 20 Consider Example 4 on p. 54.

$$a + b\mathrm{i} = \sqrt{(10 + 24\mathrm{i})} = \sqrt{2}(5 + 12\mathrm{i})^{1/2}$$

$$= \sqrt{13}\,.\sqrt{2}(\cos\alpha + \mathrm{i}\sin\alpha)^{1/2}, \quad \text{where } \tan\alpha = \tfrac{12}{5}$$

$$= \sqrt{26}(\cos\tfrac{1}{2}\alpha + \mathrm{i}\sin\tfrac{1}{2}\alpha)$$

$$2\cos^2\tfrac{1}{2}\alpha = \cos\alpha + 1 = \tfrac{5}{13} + 1, \quad 2\sin^2\tfrac{1}{2}\alpha = 1 - \cos\alpha = 1 - \tfrac{5}{13}.$$

Hence $a = 3\sqrt{2}$, $b = 2\sqrt{2}$ as before.

This result enables us to solve in complex algebra any equation of the form $z^n = w$. If $w \equiv [r, \theta]$,

$$z = w^{1/n} \equiv [r^{1/n}, (\theta + 2k\pi)/n], \quad k = 0, 1, 2, \ldots, (n-1).$$

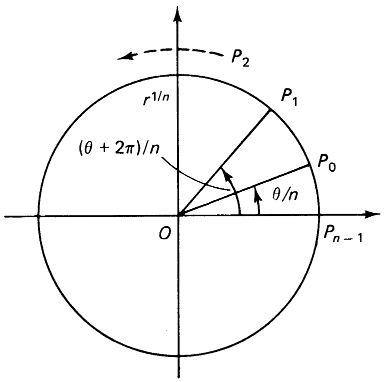

Fig. 4.9

The roots appear on an Argand diagram as points on the circle with centre O and radius $r^{1/n}$ (Fig. 4.9).

Example 21 Find the complex cube roots of 1.

Method (i): As above,

$$\begin{aligned} z^3 = 1 \quad &\Leftrightarrow \quad z^3 = \cos 0 + \mathrm{i}\sin 0 \\ &\Leftrightarrow \quad z = \cos(2k\pi/3) + \mathrm{i}\sin(2k\pi/3), \quad k = 0, 1, 2, \\ &\Leftrightarrow \quad z = [1, 0], [1, 2\pi/3], [1, 4\pi/3]. \end{aligned}$$

Method (ii): $z^3 - 1 = 0 \quad \Leftrightarrow \quad (z-1)(z^2+z+1) = 0$ (by the factor theorem)

$$\Leftrightarrow \quad z = 1 \quad \text{or} \quad \tfrac{1}{2}(-1 \pm \sqrt{3}\mathrm{i}).$$

The cube roots of 1 are often denoted by $1, \omega, \omega^2$; since $(\omega^2)^2 = \omega^4 = \omega \,.\, \omega^3 = \omega$, each complex root is the square of the other, and it is immaterial which is denoted by ω.

Exercise 4.9

1 Given that $\omega^3 = 1$, but ω is not 1, prove that

$$a^2 - ab + b^2 = (\omega a + \omega^2 b)(\omega^2 a + \omega b).$$

2 Use de Moivre's theorem to simplify

$$(\sqrt{3} + \mathrm{i})^{10} - (\sqrt{3} - \mathrm{i})^{10}.$$

3 (a) Without using tables or a calculator, simplify

$$\frac{\left(\cos\frac{\pi}{9} + \mathrm{i}\sin\frac{\pi}{9}\right)^4}{\left(\cos\frac{\pi}{9} - \mathrm{i}\sin\frac{\pi}{9}\right)^5}.$$

(b) Express z_1, where $z_1 = \dfrac{7 + 4\mathrm{i}}{3 - 2\mathrm{i}}$, in the form $p + q\mathrm{i}$, $p, q \in \mathbb{R}$.

Sketch on an Argand diagram the locus of points representing complex numbers z such that $|z - z_1| = \sqrt{5}$. Find the greatest value of $|z|$ subject to this condition.

4 The complex number z is given by $z = t + 1/t$ and $t = r(\cos\theta + \mathrm{i}\sin\theta)$. Find the cartesian equation of the locus of the point P which represents z on the Argand diagram when

(a) $r = 2$ and θ varies, (b) $\theta = \pi/4$ and r varies.

5 Show that $\cos(\pi/5) + \mathrm{i}\sin(\pi/5)$ is a root of the equation

$$z^5 + 1 = 0$$

and determine the other roots of this equation. Indicate, on an Argand diagram, the points which represent all these roots.

Express as a real number the product of the non-real roots of the equation $z^5 + 1 = 0$.

6 On the Argand diagram the fixed points A and B represent the complex numbers z_1 and z_2 respectively, where $z_1 = 2(\cos \pi/3 + i \sin \pi/3)$ and $z_2 = -2$, and the variable point P represents the complex number z. Show the positions of A and B on the Argand diagram and find
(a) the modulus and argument of $(z_1 - z_2)$,
(b) the cartesian equation of the locus of P when $\arg(z - z_1) = 2\pi/3$. Show the locus of P on the Argand diagram.

7 (a) Sketch on an Argand diagram the locus of points z satisfying the equation $|z - 1| = 1$. Shade on your diagram the region for which $|z - 1| < 1$ and

$$\pi/6 < \arg z < \pi/3.$$

(b) Show that the points on the Argand diagram representing the solutions of the equation $z^8 = 256$ lie at the vertices of a regular octagon.

8 Express $1 + i\sqrt{3}$ and $1 - i\sqrt{3}$ in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Hence evaluate

$$\left(\frac{1 + i\sqrt{3}}{1 - i\sqrt{3}}\right)^{20},$$

expressing your result in the form $a + ib$, where $a, b \in \mathbb{R}$.

9 (a) Find all the solutions of the equation $z^5 = 1$. Deduce that

$$\frac{z^5 - 1}{z - 1} = \left(z^2 - 2z\cos\frac{2\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{5} + 1\right).$$

(b) Given that $z = \cos\frac{\pi}{n} + i\sin\frac{\pi}{n}$, where n is a positive integer, $n \geqslant 2$, show that $(1 + z)^n$ is purely imaginary.

Miscellaneous exercise 4

1 (a) Given that $z_1 = 3 + 4i$ and $z_2 = -1 + 2i$, represent $z_1, z_2, (z_1 + z_2)$ and $(z_2 - z_1)$ by vectors on the Argand diagram. Express $(z_1 + z_2)/(z_2 - z_1)$ in the form $a + ib$, where $a, b \in \mathbb{R}$. Find the angle between the vectors representing $(z_1 + z_2)$ and $(z_2 - z_1)$.
(b) One root of the equation

$$z^3 - 6z^2 + 13z + a = 0,$$

where $a \in \mathbb{R}$, is $z = 2 + i$. Find the other roots and the value of a.

2 (a) Given that $z = 3 - 4i$ and $w = 12 + 5i$, express wz and w/z in the form $a + bi$, $a, b \in \mathbb{R}$.
(b) Find the modulus and argument of $1 + i\sqrt{3}$ and hence simplify $(1 + i\sqrt{3})^{10}$.
(c) Illustrate on an Argand diagram the set of complex numbers A, where

$$A = \{z: |z + 6| = 2|z - 3i|\}.$$

3 (a) By expressing $\sqrt{3} - i$ in modulus–argument form, find the least positive integer n such that $(\sqrt{3} - i)^n$ is real and positive.
(b) The point P in the Argand diagram lies outside or on the circle of radius 4 with centre at $(-1, -1)$. Write down in modulus form an inequality satisfied by the complex number z represented by the point P.

4 (a) Given that $z = 5 - 12i$, express z^{-1} and the two values of $z^{1/2}$ in the form $a + bi$, where $a, b \in \mathbb{R}$.

(b) Express $\sqrt{3} - \mathrm{i}$ in the form $r(\cos\theta + \mathrm{i}\sin\theta)$. Hence, or otherwise, show that $(\sqrt{3} - \mathrm{i})^9$ can be expressed as $c\mathrm{i}$, where $c \in \mathbb{R}$, and give the value of c.
(c) Shade in an Argand diagram the region for which

$$|z - 2| > |z|.$$

5 (a) Given that $3 + 4\mathrm{i} = (x + \mathrm{i}y)^2$, where $x, y \in \mathbb{R}$, find x and y. Find also the square roots of i in the form $a + \mathrm{i}b$, where $a, b \in \mathbb{R}$.
(b) Show that $1 + \mathrm{i}$ is a root of the equation

$$z^3 - 4z^2 + 6z - 4 = 0$$

and find the other two roots of this equation.

6 Shade on an Argand diagram the region R for which $|z| < 1$. If z is any point in the region R, and z^* is the complex conjugate of z, find the corresponding regions for w when
(a) $w = z + 3 + 4\mathrm{i}$, (b) $|wz| = 1$, (c) $w = zz^*$.

7 (a) Given that $(1 + \mathrm{i})^n = x + \mathrm{i}y$, where $x, y \in \mathbb{R}$ and $n \in \mathbb{Z}^+$, prove that $x^2 + y^2 = 2^n$.
(b) Given that $\left|\dfrac{z-1}{z+1}\right| = 2$, find a cartesian equation of the locus of z and represent the locus by a sketch on the Argand diagram. Shade the region for which the inequalities $\left|\dfrac{z-1}{z+1}\right| > 2$ and $0 < \arg z < 3\pi/4$ are both satisfied.

8 (a) If $z = 1 + \mathrm{i}\sqrt{3}$, prove that

$$z^{14} = 2^{13}(-1 + \mathrm{i}\sqrt{3}).$$

(b) On an Argand diagram, the origin and the point representing the complex number $(1 + \mathrm{i})$ form two vertices of an equilateral triangle. Find the complex number represented by the third vertex, given that its real part is positive.

9 Prove the results:
(a) $zz^* = |z|^2$,
(b) $z + z^* = 2\,\mathrm{Re}(z)$,
(c) $\left(\dfrac{z_1}{z_2}\right)^* = \dfrac{z_1^*}{z_2^*}$.

Given that $|z| = 1$ and $w = \dfrac{1+z}{1-z}$, prove that $\mathrm{Re}(w) = 0$.

Indicate on separate Argand diagrams the locus of the points z for which $|z| = 1$ and and locus of the points w for which $\mathrm{Re}(w) = 0$. Mark in the point $z = \cos\frac{1}{3}\pi + \mathrm{i}\sin\frac{1}{3}\pi$ and the position of the corresponding point w.

10 (a) If $z_1 = (\cos\pi/4 + \mathrm{i}\sin\pi/4)$ and $z_2 = (\cos\pi/3 - \mathrm{i}\sin\pi/3)$, find
(i) the arguments of $z_1z_2^2$ and z_1^3/z_2;
(ii) the real and imaginary parts of $z_1^2 + \mathrm{i}z_2$;
(iii) the moduli and arguments of the three cube roots of z_2.
(b) Indicate on an Argand diagram the locus of z if
(i) $|z| = |z - 4|$;
(ii) $[\mathrm{Re}(z)]^2 = \mathrm{Im}(z^2)$,
(iii) $2|z|^2 + \mathrm{Re}(z^2) = 1$.

11 Describe the configuration of points on the Argand diagram which represent complex numbers w where $|w + 1 - \mathrm{i}| \leqslant 1$. Explain why no such complex number exists with $\arg w = \frac{1}{4}\pi$.

Find, if possible, complex numbers z satisfying

$$|z + 1 - i| = |z - 2 - 4i|$$

which (a) are real, (b) are purely imaginary, (c) have argument $\frac{1}{4}\pi$, (d) have argument $-\frac{1}{4}\pi$.

12 Describe the locus defined by each of the following equations, and illustrate each locus on an Argand diagram.
(a) $|z + 1|^2 + |z - 1|^2 = 4$,
(b) $|z + i| + |z - i| = 3$,
(c) $\arg(z - 1) = \arg(z + 1)$.

13 (a) Show shaded on separate Argand diagrams the regions for which

$$|z - 4| > |z|, \quad |2z + 3| \geqslant 4.$$

(b) Sketch on an Argand diagram the locus of a point P representing the complex number z, where

$$|z - 1| = |z - 3i|,$$

and find z when $|z|$ has its least value on this locus.
(c) Shade on an Argand diagram the region representing all complex numbers z which satisfy both

$$1 \leqslant |z| \leqslant 2 \quad \text{and} \quad \pi/3 \leqslant \arg z \leqslant \pi.$$

14 (a) Show that $z = 1 - i$ is a root of the equation

$$z^4 + 3z^2 - 6z + 10 = 0.$$

Find the other three roots of this equation.
(b) Sketch the curve on the Argand diagram defined by

$$|z - 1| = 1, \quad \text{Im}(z) \geqslant 0.$$

Find the value of z at the point P in which this curve is cut by the line $|z - 1| = |z - 2|$. Find also the values of $\arg z$ and $\arg(z - 2)$ at P.

Answers

Exercise 1.1

1 $14x - 13y = 0$
2 $x - y = 0, x - y = 6$
3 (a) $-3/8$; (b) 3; (c) 8/9
4 (a) $3y = x + 1$; (b) $(7, \frac{8}{3})$;
(c) $(6, 4)$, 5; (d) $\frac{50}{3}$
5 (a) $(12, 9)$; (b) $3y + 4x = 75$;
(c) $\left(9\frac{3}{8}, 12\frac{1}{2}\right)$; (d) $\left(21\frac{3}{8}, 21\frac{1}{2}\right)$
6 (a) $(11, 10)$; (c) $2:1$;
(d) $(8, 5)$, $4y = x + 12$
7 (a) $(5, 3)$; (b) $5x + 2y = 13$;
(c) $(2, 0)$; (d) $x + 2y = 9$

Exercise 1.3

1 $x\cos\theta + y\sin\theta = 2 + 2\cos\theta$;
$(-2, 0)$; $x^2 + y^2 = 2x + 2$
2 8 units^2; $x = 0, y = \frac{3}{4}x$
3 (a) $x^2 + y^2 - 2x - 24 = 0$;
(b) $(8/3, 0)$;
(c) $x^2 + y^2 - 2x + 5y - 24 = 0$
4 (a) $x^2 + y^2 - 6x - 6y + 5 = 0$;
(b) $3x^2 + 3y^2 - 6x - 26y + 3 = 0$
5 $x^2 + y^2 - 10x - 8y + 16 = 0$; $x = 0$,
$40y + 9x = 0$
6 $(x + 3)^2 + (y - 6)^2 = 36$;
$4x - 3y = 0$
7 $x(x - 4) + y(y - 2) = 0$;
$2x + y = 5$
8 $y + 2x = 5$; $(10, -15)$; $(5, -5)$
9 $x\cos\theta + y\sin\theta = a$; $(a(2\cos\theta + 1)$,
$a(2\sin^2\theta - 1 - \cos\theta)/\sin\theta)$;
$(0, -1{\cdot}155a)$, $(0, 1{\cdot}155a)$
10 $(12/5, 9/5)$;
$x\left(x - \frac{6}{5}\right) + y\left(y - \frac{17}{5}\right) = 0$
11 (a) $x^2 + y^2 - 6x - 6y + 5 = 0$;
(b) $3x^2 + 3y^2 - 6x - 26y + 3 = 0$
12 $(x + 2)(x - 3) + (y - 1)(y + 2) = 0$ *or* $x^2 + y^2 - x + y = 8$; a circle with $(3, -2)$ and $(-2, 1)$ at the ends of a diameter

Miscellaneous Exercise 1

1 (a) $7x - 6y - 20 = 0$;
(b) $2x - y + 1 = 0$;
(c) $28x + 17y - 16 = 0$
2 $-\frac{3}{2}, -\frac{15}{8}$; $x + y - 3 = 0$,
$x + 7y - 9 = 0$
3 (a) $3x + 5y - 8 = 0$;
(b) $8x^2 + 8y^2 + 10x - 104y + 313 = 0$;
(c) $y^2 = 12(x - 1)$
4 (i) (a) $3x - 2ty + 3(1 + t^2) = 0$;
(b) $y^2 = 9(x + 1)$;
(ii) (a) (b) $x - 2y + 11 = 0$;
(iii) (a) $3tx - y - 2t^3 = 0$,
(b) $y^2 = 2x^3$; (iv) (a) (b) $x - 2y - 1 = 0$
Note in (ii) and (iv) the curve is a straight line.
6 $x^2 + y^2 - 6x - 16 = 0$; $32{\cdot}36$ units^2
7 $x + 2y - 11 = 0$, $x - y + 1 = 0$;
$\tan^{-1}\left(\frac{9}{8}\right)$; $\left(\frac{11}{3}, \frac{7}{3}\right)$
8 $8\frac{2}{3}$ units^2
9 $hx/a^2 + ky/b^2 = 1$;
$[a^2(b - k)/(bh - ak)$,
$-b^2(a - h)/(bh - ak)]$
10 $(3 + 2\sqrt{3}, -1 - \sqrt{3})$,
$(3 - 2\sqrt{3}, -1 + \sqrt{3})$
11 $2x + y - 10 = 0$, $x - 2y + 5 = 0$;
$15\frac{3}{5}$ units^2
12 10 and 13
13 $(9/4, -5/2)$
14 $p = 1/5$, $q = 3/5$; centre $(2, 1)$, radius $\sqrt{5}$; $x/(ma) + y/(nb) = 1$;
$[m(n - 1)a/(n - m), -n(m - 1)b/(n - m)]$
16 $c = 2k/\sqrt{5}$

Exercise 2.2

3 (a) $2x - y + 2a = 0$; (b) $y^2 = 8ax$
7 $(2ap^2, 3ap)$; $4y + 6px = 12ap + 15ap^3$,
$3ap^2/\sqrt{(16 + 36p^2)}$
10 (b) $(-\frac{1}{4}ap^2, \frac{1}{2}ap)$; (d) $y^2 = -ax$
12 $(2at^2, 3at)$
13 $9y^2 = 4a(3x - 2a)$
14 $2y^2 + ax = 0$; $(a, 0)$, $(-a/8, 0)$

17 $y - tx + at + at^2 = 0$,
$ty + x - a - 2at - at^3 = 0$

Exercise 2.3
9 ab

Exercise 2.5
3 $a^2y^2 - b^2x^2 = 4x^2y^2$
4 $[a(\sec\theta + \tan\theta), b(\sec\theta + \tan\theta)]$,
$[a(\sec\theta - \tan\theta), -b(\sec\theta - \tan\theta)]$; ab
8 $[(a^2\cos\alpha)/p, -(b^2\sin\alpha)/p]$;
$3\sqrt{2}x \pm \sqrt{7}y = \pm 15$
(all four combinations)
9 $2\pi/3$
10 $x^2 + y^2 = 5$
13 $y + 4x = \pm 4c$; $8c/\sqrt{17}$
19 (a) $-1/(t_1t_2)$

Exercise 2.6
1 2θ
2 (a) $3x + y = 0, x - 3y = 0$;
(b) $8:1:-3$;
(c) $4alx^2 + 4amxy - y^2 = 0$
3 $x^2 - y^2 + 2xy = 0$

Miscellaneous Exercise 2
1 $[apq, a(p+q)]$
2 $bl^2 = mn$
4 $\frac{3}{4}$
10 $[apq, a(p+q)]$; $y^2 = 4a(x+a)$
14 (b) That part of the ellipse
$x^2/64 + y^2/16 = 1$
which lies in the first quadrant
15 $3a/2$
19 $b^2x^2 + a^2y^2 = a^2b^2(lx + my)^2$:
$x^2 + y^2 = a^2b^2/(b^2 - a^2)$
20 $(x^2 + y^2)^2 = 4c^2xy$
21 $ty - t^3x = c - ct^4$
22 $x^2 + y^2 = a^2 + b^2 - k^2$
23 $[2ct_1t_2/(t_1 + t_2), -2c{t_1}^2{t_2}^2/(t_1 + t_2)]$
25 $2x - y(t_1 + t_2) + 2at_1t_2 = 0$;
$x^2 + y^2 - a({t_1}^2 + {t_2}^2)x - 2a(t_1 + t_2)y +$
$a^2t_1t_2(t_1t_2 + 4) = 0$
27 $3x - y + 6c = 0$;
$3(x^2 + y^2) - 16cy - 22c^2 = 0$

Exercise 3.2
1 $r = a\operatorname{cosec}\theta$
2 $r\cos\theta = a$ for $-\pi/2 < \theta < \pi/2$;
$r\cos\theta = -a$ for $-\pi \leqslant \theta < -\pi/2$ and
$\pi/2 < \theta \leqslant \pi$
3 $r(\cos\theta - \sin\theta) = -a$ or
$r\cos(\theta + \pi/4) = -a/\sqrt{2}$
4 $r\cos(\theta + \pi/3) = \dfrac{1}{2}a$
5 $r = 2a\cos\theta$
6 $r = -2a\cos\theta$
7 $a/r = 1 + \cos\theta$
8 $(x^2 + y^2)^2 = a^2(x^2 - y^2)$
9 $y^2 = a(a - 2x)$
10 $a^2(x^2 + y^2) = (x^2 + y^2 - 2ax)^2$
11 $x\cos\alpha + y\sin\alpha = a$
12 $r = 2a\cos\theta$
13 $r = 2a\sin\theta$
14 $r^2 = a^2\sin 2\theta$
15 $r^2\cos 2\theta = a^2$
16 $r^2\sin 2\theta = a^2$
19 $(a, 2\pi/3), (a, -2\pi/3)$
22 $(0,0), (1,1)$
24 $(3a/2, \pi/3), (3a/2, -\pi/3)$; $\theta = \pm\pi/3$,
$r\cos\theta = 3a/4$

Exercise 3.3
1 (a) $\pi/2$; (b) $a^2/2$
2 2π; $\theta = \pm\pi/4$
4 $\pi a^2/12$
5 $x^2 - y^2 = 0, \pi/2$; 1
6 18π
7 $5\pi a^2$
8 $(\sqrt{2}, \pi/4)$; $r\sin\theta = 1, r\cos\theta = 1$,
$\theta = \pi/4$; $1 + 3\pi/2$

Exercise 3.5
1 $\mathbf{T}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$;
$\mathbf{T}_\theta = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$; $2\phi - 2\theta$
2 (a) $6:1, \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} \\ \frac{1}{6} & \frac{2}{3} \end{pmatrix}, \left(\frac{1}{3}, -\frac{1}{6}\right)$,
$x + y = 0$ and $x + 2y = 0$;
(b) $|\mathbf{T}_2| = 0, y = \frac{1}{2} - 2x, x + 2y = 0$
3 $\mathbf{R}(\alpha)$: clockwise rotation through an angle α; $\mathbf{M}(k)$: magnification by a factor k
4 $\mathbf{M} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$;
$y + 3x + 4 = 0$
5 $(0,0), \{(x,y): 3x + 5y = 0\}$,
$S_2 = \{(x,y): 2x + y = 0\}, \dfrac{1}{\sqrt{5}}$
6 $(3/4, 1)$
7 $2/\sqrt{5}$; a rotation through $\arctan(\frac{1}{2})$
8 $\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$
9 (a) $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$;

(b) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$; $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,
$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}$, $\begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{pmatrix}$
$\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}$, $(-1-\sqrt{3}, \sqrt{3}-1)$

Miscellaneous Exercise 3

1 $3\pi a^2/2$
2 $\pi a^2/8$; $\left(2a/3, \pm \sin^{-1}\dfrac{1}{\sqrt{6}}\right)$
3 $(3\pi + 8\sqrt{2} + 2)a^2/16$
4 $\pi/2 - 1$
5 $2\beta - 2\alpha$
6 $\mathbf{M} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$;
$\mathbf{M}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 1 \end{pmatrix}$
7 (c) Quarter-turn about O
8 (a) Clockwise quarter-turn about O; (b) half-turn about O, (c) identity
9 (a) Half-turn about Ox; (b) reflection in the plane $y = 0$; (c) quarter-turn about Oz
10 $(0,0), (2,0), (1,2)$; $\dfrac{1}{4}\begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$
11 Reflection in $(2, -1)$
12 Projection onto $2y = 3x$

Exercise 4.2

1 $z_1 = -\frac{1}{2}(1+i), z_2 = -1 + i$
2 (a) $2 - 11i$; (b) $4 + i$
3 $-\frac{6}{5} + \frac{12}{5}i$
4 $\frac{4}{5} - \frac{3}{5}i$
5 $x = 1, y = -\frac{1}{2}$
6 (a) 8
7 $\frac{1}{5} - \frac{3}{5}i$

Exercise 4.6

1 $2 - i, -4$
2 (a) $13, -23°$; (b) $1, 90°$
3 $2 + i$, $1 + 2i$; the coefficients of the equation are not real
4 $1; \dfrac{2\pi}{3}$; $z = 1$ or $-\frac{1}{2} \pm \dfrac{i\sqrt{3}}{2}$
5 $\pm(1 \pm 2i)$; $\sqrt{5}$
6 $|z_1| = 5\sqrt{2}$, $|z_2| = 5$, $\arg(z_1/z_2) = \frac{1}{4}\pi$
7 $p = \sqrt{2}$, $q = -2(1 + \sqrt{2})$; $p = -\sqrt{2}$, $q = 2(\sqrt{2} - 1)$
8 $1 \pm 3i$, $-1 \pm i$

Exercise 4.7

1 (a) $|z| = 2$, $|z^5| = 32$, $\arg z = \pi/3$, $\arg(z^5) = -\pi/3$, $\operatorname{Im}(z^5) = -16\sqrt{3}$; (b) $2 + 3i$
2 $|z - 2 - i| = \sqrt{5}$
4 (a) 36; (b) -4; (c) $(-7 - 4\sqrt{2}i)/9$
6 5
7 Radius 5/2, centre $-3/2 + 2i$
8 $(1 + i\sqrt{3})/2$
9 (b) Modulus 3, arguments $\pm\pi/3$, π
10 (a) Circle centre $1 + 0i$, radius 3; (b) the perpendicular bisector (mediator) of the line joining $2 + 0i$ to $0 - i$ (i.e. $y + 2x = 3/2$); (c) the circle $3(x^2 + y^2) + 10x + 3 = 0$; (d) the parabola $y^2 = 2x - 1$
13 (a) The line $x = \frac{1}{2}$; (b) the ellipse with foci $(2,0)$, $(-1,0)$ and eccentricity $\frac{3}{4}$; (c) an arc of a circle, above the real axis, at which the points $(2,0)$ and $(-1,0)$ subtend an angle of $\pi/4$

Exercise 4.9

2 $-2^{10}\sqrt{3}i$
3 (a) -1; (b) $1 + 2i$, $2\sqrt{5}$
4 (a) $x^2/25 + y^2/9 = 1/4$; (b) $x^2 - y^2 = 2$
5 $\operatorname{cis}(3\pi/5)$, -1, $\operatorname{cis}(7\pi/5)$, $\operatorname{cis}(9\pi/5)$; 1
6 (a) $2\sqrt{3}$, $\pi/6$; (b) the half-line starting at A and making an angle $2\pi/3$ with Ox
8 $2(\cos\pi/3 \pm i\sin\pi/3)$; $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$
9 (a) 1, $(\cos 2\pi/5 \pm i\sin 2\pi/5)$, $(\cos 4\pi/5 \pm i \sin 4\pi/5)$

Miscellaneous Exercise 4

1 (a) $-(1 + i)$; 135°, (b) $2 - i$, 2; -10
2 (a) $56 - 33i$, $(16 + 63i)/25$; (b) $2, \pi/3, -512(1 + i\sqrt{3})$
3 (a) 12; (b) $|z + 1 + i| \geqslant 4$
4 (a) $(5 + 12i)/169$, $\pm(3 - 2i)$; (b) 512
5 (a) 2, 1 and -2, -1, $\pm(1 + i)/\sqrt{2}$; (b) $1 - i$, 2
7 (b) $3x^2 + 3y^2 + 10x + 3 = 0$
8 (b) $\frac{1}{2}[(1 + \sqrt{3}) + i(1 - \sqrt{3})]$
10 (a) (i) $-5\pi/12$, $13\pi/12$(or $-11\pi/12$); (ii) $\frac{\sqrt{3}}{2} + \frac{3}{2}i$; (iii) modulus 1, arguments $-\pi/9$, $5\pi/9$, $11\pi/9$

11 The circle, with centre $-1 + i$ and radius 1, and its interior; all points in the region have arguments between $\pi/2$ and π inclusive;
(a) 3; (b) 3i; (c) $3(1 + i)/2$;
(d) none

12 (a) The circle with $(1 + 0i)$, $(-1 + 0i)$ at the ends of a diameter;
(b) the ellipse with foci i, $-i$ and eccentricity 2/3;
(c) the real axis excluding the part between $(1, 0)$ and $(-1, 0)$

13 (b) $(-2 + 6i)/5$

14 (a) $1 + i$, $-1 \pm 2i$;
(b) $(3 + i\sqrt{3})/2$; 30°, 120°

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